

RIESZ TRANSFORMS AND SPECTRAL MULTIPLIERS OF THE HODGE-LAGUERRE OPERATOR

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ABSTRACT. On \mathbb{R}_+^d , endowed with the Laguerre probability measure μ_α , we define a Hodge-Laguerre operator $\mathbb{L}_\alpha = \delta\delta^* + \delta^*\delta$ acting on differential forms. Here δ is the Laguerre exterior differentiation operator, defined as the classical exterior differential, except that the partial derivatives ∂_{x_i} are replaced by the “Laguerre derivatives” $\sqrt{x_i}\partial_{x_i}$, and δ^* is the adjoint of δ with respect to inner product on forms defined by the Euclidean structure and the Laguerre measure μ_α . We prove dimension-free bounds on L^p , $1 < p < \infty$, for the Riesz transforms $\delta\mathbb{L}_\alpha^{-1/2}$ and $\delta^*\mathbb{L}_\alpha^{-1/2}$. As applications we prove the strong Hodge-de Rahm-Kodaira decomposition for forms in L^p and deduce existence and regularity results for the solutions of the Hodge and de Rham equations in L^p . We also prove that for suitable functions m the operator $m(\mathbb{L}_\alpha)$ is bounded on L^p , $1 < p < \infty$.

CONTENTS

1. Introduction	2
2. Laguerre operators on functions	7
2.1. The operator \mathcal{L}_α	7
2.2. The operators $\mathcal{L}_{\alpha,I}$	9
3. The Hodge-Laguerre operator	12
3.1. Differential forms on \mathbb{R}_+^d	12
3.2. The Hodge-Laguerre operator on forms	14
3.3. The diagonalization of the Hodge-Laguerre operator	17
3.4. A self-adjoint extension of the Hodge-Laguerre operator	19
4. Hodge-De Rahm-Kodaira decomposition and Riesz-Laguerre transforms	26
5. The Bilinear Embedding Theorem and its applications	27
5.1. Riesz-Laguerre transforms on $L^p(\mathbb{R}_+^d, \mu_\alpha; \wedge^r)$	28
5.2. The Hodge decomposition for $L^p(\mathbb{R}_+^d, \mu_\alpha; \wedge^r)$	29
5.3. The Hodge system and the de Rham equation in L^p	31
6. Bellman function	32
7. Proof of the Bilinear Embedding Theorem	39
8. Spectral multipliers	45
References	48

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1. INTRODUCTION

For every multiindex $\alpha = (\alpha_1, \dots, \alpha_d) \in (-1, \infty)^d$ denote by μ_α the *Laguerre measure* on \mathbb{R}_+^d , i.e. the probability measure with density

$$\rho_\alpha(x) = \prod_{i=1}^d \frac{x_i^{\alpha_i} e^{-x_i}}{\Gamma(\alpha_i + 1)} dx_i$$

with respect to the Lebesgue measure on \mathbb{R}_+^d .

The *Laguerre operator* \mathcal{L}_α is the differential operator on \mathbb{R}_+^d defined by

$$\mathcal{L}_\alpha = - \sum_{i=1}^d \left[x_i \frac{\partial^2}{\partial x_i^2} + (\alpha_i + 1 - x_i) \frac{\partial}{\partial x_i} \right].$$

The operator \mathcal{L}_α , considered as an operator on the Hilbert space $L^2(\mathbb{R}_+^d, \mu_\alpha)$ with domain the space of smooth functions with compact support $C_c^\infty(\mathbb{R}_+^d)$, is symmetric and nonnegative for all $\alpha \in (-1, \infty)^d$. If $\alpha_i \in (-1, 1)$ for some $i \in \{1, 2, \dots, d\}$ then it has several self-adjoint extensions, depending on the boundary condition at $\{x \in \mathbb{R}_+^d : x_i = 0\}$. For $\alpha \in [1, \infty)^d$ it is essentially self-adjoint and so it has a unique self-adjoint extension whose spectral resolution is given by the orthogonal system of generalized Laguerre polynomials $\{L_k^\alpha : k \in \mathbb{N}\}$ in d variables. The Laguerre orthogonal system provides also the spectral resolution of an extension of \mathcal{L}_α for $\alpha \notin [1, \infty)^d$ that satisfies an appropriate boundary condition at $x_i = 0$ whenever $\alpha_i < 1$ [40].

Henceforth, for all $\alpha \in (-1, \infty)^d$ we shall denote by \mathcal{L}_α the self-adjoint realization of the Laguerre operator that has the Laguerre polynomials as eigenfunctions. It is well known that for $\alpha \in [-1, \infty)^d$ the operator \mathcal{L}_α generates a symmetric diffusion semigroup on the measure space $(\mathbb{R}_+^d, \mu_\alpha)$, called the *Laguerre semigroup*. In his pioneering work [36], E. M. Stein realized that many classical results concerning the heat and the Poisson semigroups associated to the Laplacian on the Euclidean setting generalize to the more abstract setting of symmetric diffusion semigroups. This was the starting point of the so called harmonic analysis of semigroups, which centers around the study of the boundedness properties on the Lebesgue spaces L^p of various operators naturally associated to the semigroup and its generator, such as maximal functions, Littlewood-Paley functions, spectral multipliers and Riesz transforms. Some milestones in this abstract theory are the works of M. Cowling [8], of R. R. Coifman, R. Rochberg and G. Weiss [7] and, quite recently, the paper of A. Carbonaro and O. Dragičević [5] on spectral multipliers for symmetric contraction semigroups.

Even though some of these general results are optimal, sharper results can be obtained for particular subclasses of symmetric contraction semigroups, and there is a vast body of literature concerning specific semigroups such as, for instance, the semigroups generated by invariant Laplacians or sublaplacians on Lie groups, by the Laplace-Beltrami operator on Riemannian manifolds, the Ornstein-Uhlenbeck semigroup, both in the finite and in the infinite dimensional setting, and the semigroups associated to various orthogonal systems of polynomials. In the last class falls the Laguerre semigroup, whose harmonic analysis has been investigated recently by various people. Specifically we mention the work of B. Muckenhoupt in the one-dimensional setting [26, 27] and, in higher dimension, those of U. Dinger on

the weak-type estimate for the maximal function [10], of E. Sasso on spectral multipliers [33, 32] and on the maximal operator for the holomorphic semigroup [34] of E. Sasso [35] and of L. Forzani, E. Sasso and R. Scotto on the weak type inequality for the Riesz-Laguerre transforms [13, 12], of P. Graczyk *et al.* on higher order Riesz transforms [15]. Particularly relevant for the results of this paper are the papers of A. Nowak [30] and of A. Nowak and K. Stempak [31] on Riesz transforms and on L^p -contractivity of the Laguerre semigroup.

In this paper we investigate the L^p -boundedness of the Riesz transforms and spectral multipliers for the Hodge-Laguerre operator \mathbb{L}_α , a generalisation to differential forms of the Laguerre operator \mathcal{L}_α on functions. We recall that, if M is a Riemannian manifold, the Hodge-de Rham operator on differential forms is the operator $\square = dd^* + d^*d$, where d denotes the exterior differentiation operator mapping r -forms to $(r+1)$ -forms and d^* is its adjoint with respect to the inner product on forms defined by the the Riemannian structure and the Riemannian measure. We define the Hodge-Laguerre operator acting on differential forms on \mathbb{R}_+^d as $\mathbb{L}_\alpha = \delta\delta^* + \delta^*\delta$, where δ is the Laguerre exterior differentiation operator, defined much as the classical exterior differential, except that the partial derivatives ∂_{x_i} are replaced by the “Laguerre derivatives” $\sqrt{x_i}\partial_{x_i}$, and δ^* is the adjoint of δ with respect to inner product on forms defined by the Euclidean structure and the Laguerre measure μ_α (see Section 2.1 for more details).

On manifolds, the Riesz transforms on forms are the operators $\mathcal{R} = d\square^{-1/2}$, mapping r -forms to $(r+1)$ -forms, and its formal adjoint $\mathcal{R}^* = \square^{-1/2}d^*$, mapping r -forms to $(r-1)$ -forms.

In [37] R. Strichartz proved the on a *complete* Riemannian manifold the Hodge operator is essentially self-adjoint. He also proved that the Riesz transforms $d\square^{-1/2}$ and $\square^{-1/2}d^*$ are bounded on L^2 . It is well known that there is a connection between the boundedness of Riesz transforms on L^2 -forms and the L^2 -Hodge-de Rham decomposition of the space $L^2(M, \Lambda^r)$ of square-integrable r -forms as the direct orthogonal sum

$$L^2(M, \Lambda^r) = \ker_r(\square) \oplus \overline{d(C_c^\infty(M; \Lambda^{r-1}))} \oplus \overline{d^*(C_c^\infty(M; \Lambda^{r+1}))}$$

where

- (i) $\ker_r(\square)$ is the kernel of \square in $L^2(M, \Lambda^r)$,
- (ii) $\overline{d(C_c^\infty(M; \Lambda^{r-1}))}$ is the closure in $L^2(M, \Lambda^r)$ of the image of d on the space $C_c^\infty(M; \Lambda^{r-1})$ of smooth forms with compact support,
- (iii) $\overline{d^*(C_c^\infty(M; \Lambda^{r+1}))}$ is the closure in $L^2(M, \Lambda^r)$ of the image of d^* on the space $C_c^\infty(M, \Lambda^{r+1})$,

(see [9, 17]). Indeed the operators $\mathcal{R}\mathcal{R}^*$ and $\mathcal{R}^*\mathcal{R}$ are precisely the orthogonal projections onto the spaces $\overline{d(C_c^\infty(M; \Lambda^{r-1}))}$ and $\overline{d^*(C_c^\infty(M; \Lambda^{r+1}))}$, respectively. To prove the existence of an analogue of the Hodge decomposition for L^p forms when $p \neq 2$, various authors have investigated the boundedness of the Riesz transforms on L^p , under suitable geometric conditions on the manifold [24, 23, 22, 21, 20].

On 0-forms the operator \square reduces to $d^*d = \Delta$, the Laplace-Beltrami operator on functions, and $d^* = 0$. Thus $\mathcal{R}^* = 0$, and $\mathcal{R} = d\Delta^{-1/2}$ is the Riesz transform mapping functions to 1-forms, a singular integral whose boundedness on $L^p(M)$ has been extensively investigated by many authors [2, 1, 6, 4].

In the Laguerre context, the boundedness on $L^p(\mathbb{R}_+^d, \mu_\alpha)$, $1 < p < \infty$, of the scalar Riesz transforms $\delta\mathcal{L}_\alpha^{-1/2}$, $i = 1, \dots, d$ on functions has been proved by A.

Nowak when $\alpha \in [-1/2, \infty)^d$, using a Littlewood-Paley-Stein square function [30]. The estimates obtained by Nowak are independent of the dimension. Quite recently B. Wróbel has described a general scheme for deducing dimension free L^p estimates of d -dimensional Riesz transforms from the boundedness of one-dimensional Riesz transforms [39]. By combining his result with the one-dimensional estimate of B. Muckenhoupt [27], Wróbel obtains dimension independent estimates of the scalar Riesz transforms $\delta_i \mathcal{L}_\alpha^{-1/2}$, $i = 1, \dots, d$, for all $\alpha \in (-1, \infty)^d$. To the best of our knowledge, no dimension-free estimates are known for the vector valued Riesz transform $\delta \mathcal{L}_\alpha^{-1/2}$.

Our first result is that for $\alpha \in [-1/2, \infty)^d$ the Riesz transforms associated to the Hodge-Laguerre operator, $\delta \mathbb{L}_\alpha^{-1/2}$ and $\mathbb{L}_\alpha^{-1/2} \delta^*$, are bounded from $L^p(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ to $L^p(\mathbb{R}_+^d, \mu_\alpha; \Lambda^{r+1})$ and to $L^p(\mathbb{R}_+^d, \mu_\alpha; \Lambda^{r-1})$ respectively, for all $p \in (1, \infty)$ and all $r \in \{1, 2, \dots, d\}$. We emphasise the fact that our bounds are independent of the dimension d and of the multi index α . When $r = 0$ the analogous result holds for the Riesz transform $\delta \mathbb{L}_\alpha^{-1/2} = \delta \mathcal{L}_\alpha^{-1/2}$ provided that the domain is restricted to the forms in $L^p(\mathbb{R}_+^d, \mu_\alpha; \Lambda^0)$ with integral zero. Actually, we shall prove the stronger result that for every $\rho \leq r/2$ the shifted Riesz transforms $\delta(\mathbb{L}_\alpha - \rho I)^{-1/2}$ and $(\mathbb{L}_\alpha - \rho)^{-1/2} \delta^*$ are bounded from $L^p(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ for all $p \in (1, \infty)$, with dimension-free bounds. We shall apply this result to obtain the following strong form of the Hodge-De Rham decomposition in L^p for all $1 < p < \infty$ and $r = 1, \dots, d$

$$L^p(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r) = dW^{1,p}(\mathbb{R}_+^d, \mu_\alpha; \Lambda^{r-1}) \oplus dW^{1,p}(\mathbb{R}_+^d, \mu_\alpha; \Lambda^{r+1})$$

where $W^{1,p}(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ denotes a $(1, p)$ -Sobolev space of r -forms, defined as the domain of the closure in $L^p(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ of the operator that maps a “polynomial” form ω into the pair of forms $(\delta\omega, \delta^*\omega)$ (see Section 5.2 for more details).

A second application is to show existence theorems and L^p -estimates for the Hodge-Laguerre system and de Rham-Laguerre operator. The Hodge-Laguerre system concerns the solvability in $L^p(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ of the system of equations $\delta\omega = \alpha$ and $\delta^*\omega = \beta$ for all $\alpha \in W^{1,p}(\mathbb{R}_+^d, \mu_\alpha; \Lambda^{r+1})$ and all $\beta \in W^{1,p}(\mathbb{R}_+^d, \mu_\alpha; \Lambda^{r-1})$ such that $\delta\alpha = 0$ and $\delta^*\beta = 0$. The de Rham-Laguerre equation concerns the solvability in $L^p(\mathbb{R}_+^d, \mu_\alpha; \Lambda^{r-1})$ of the equation $\delta\omega = \alpha$ for all $\alpha \in W^{1,p}(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ such that $\delta\alpha = 0$. Our results here are the analogues in the Laguerre setting of results obtained by X.-D. Li over weighted complete Riemannian manifolds under suitable curvature and completeness assumptions [20]. Note that \mathbb{R}_+^d is not a complete manifold.

We now describe briefly the method used to prove the L^p -boundedness of Riesz transforms. We adapt to our setting Carbonaro and Dragičević’s proof of Bakry’s result on the L^p -boundedness of the Riesz transform on functions on complete Riemannian manifolds whose Ricci curvature is bounded from below [4]. They reduce the problem to a bilinear estimate involving the covariant derivatives of the Poisson semigroups acting on functions and on 1-forms. To prove the bilinear estimate they adapt the technique of Bellman functions, introduced in harmonic analysis by F. Nazarov, S. Treil and A. Volberg in [28]. A crucial role in our analysis is played by the fact that the Hodge-Laguerre operator \mathbb{L}_α on forms acts diagonally on the coefficients of the form. Namely, if $\omega = \sum_I \omega_I dx_I$ is a form in $C^\infty(\mathbb{R}_+^d, \Lambda^r)$

then

$$(1.1) \quad \mathbb{L}_\alpha \omega = \sum_I \mathcal{L}_{\alpha, I} \omega_I dx_I,$$

where the $\mathcal{L}_{\alpha, I}$ are some generalisations of the Laguerre operator \mathcal{L}_α . On 1-forms the operators $\mathcal{L}_{\alpha, \{i\}}$ coincide with the operators M_i^α introduced by Nowak in [30] and studied by Nowak and Stempak in [31] in connection with conjugate Poisson integrals.

In the last section we prove a spectral multiplier theorem for the Hodge-Laguerre operator. The Hodge-Laguerre operator on r -forms has a self-adjoint extension \mathbb{L}_α on $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ with spectral resolution

$$\mathbb{L}_\alpha = \sum_{n \geq r} n \mathcal{P}_n^\alpha,$$

where, for each integer $n \geq r$, \mathcal{P}_n is the orthogonal projection on a finite-dimensional space of “polynomial” forms. Thus, by the spectral theorem, if $m = (m_n)_{n \geq r}$ is a bounded sequence in \mathbb{C} the operator

$$m(\mathbb{L}_\alpha) = \sum_{n \geq r} m_n \mathcal{P}_n^\alpha$$

is bounded on $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ and $\|m(\mathbb{L}_\alpha)\|_{2 \rightarrow 2} = \sup_{n \geq r} |m_n|$.

We give a sufficient condition for the boundedness of $m(\mathbb{L}_\alpha)$ on $L^p(\Lambda^r(\mathbb{R}_+^d), \mu_\alpha)$ also for $p \neq 2$. This is a particular instance of the *spectral multiplier problem* for self-adjoint operators on $L^2(X, \mu)$ where (X, μ) is a σ -finite measure space, which has been investigated in a great variety of contexts in the last thirty years. Since the literature on the subject is huge, here we only cite a few landmarks which are more pertinent to our result. The pioneering work is the already cited monograph [36], where E. M. Stein proved a spectral multiplier theorem for generators of symmetric diffusion semigroups. Stein’s result was subsequently sharpened by M. Cowling in [8], who proved that if the operator $-A$ generates a symmetric contraction semigroup on (X, μ) and the function m is bounded and holomorphic in the sector

$$S_{\theta_p} = \{z \in \mathbb{C} : |\arg z| < \theta_p\}$$

where $\theta_p = \frac{\pi}{2} \left| \frac{2}{p} - 1 \right|$, for some $p \in (1, 2)$, then the operator $m(A)$ defined spectrally on $L^2(X, \mu)$ extends to a bounded operator on $L^q(X, \mu)$ for all q with $p < q < p'$.

It was known for some time that for some specific generators the angle θ_p is not optimal. In particular, if $\mathcal{L}_{OU} = -\frac{1}{2}\Delta + x\nabla$ is the generator of the Ornstein-Uhlenbeck semigroup on \mathbb{R}^n endowed with the Gauss measure γ , then J. García-Cuerva et al. [14] (see also [25]), proved that it suffices to assume that the spectral multiplier m is bounded and holomorphic in the smaller sector $S_{\phi_p^*}$ with $\phi_p^* = \arcsin \left| \frac{2}{p} - 1 \right|$ (plus some additional differential condition on the boundary of the sector) to obtain L^q boundedness of $m(\mathcal{L})$ on all L^q , with $p \leq q \leq p'$. It is noteworthy to remark that W. Hebisch, G. Mauceri and S. Meda proved that holomorphy of m in the sector $S_{\phi_p^*}$ becomes also necessary if we assume that the multiplier m is *uniform*, i.e. that all the operators $m(t\mathcal{L})$, $t > 0$, are uniformly bounded on L^p [19]. These results are particularly relevant here, because

the Ornstein-Uhlenbeck operator is strictly related to the Laguerre operator on functions via a change of variables, as remarked in [18]. Indeed, exploiting the relationship between \mathcal{L}_{OU} and \mathcal{L}_α E. Sasso in [32] proved that the results in [14, 19] hold also for the Laguerre operator \mathcal{L}_α on functions. Finally, quite recently, A. Carbonaro and O. Dragičević [5] proved that ϕ_p^* is indeed the optimal angle in a universal multiplier theorem for generators of symmetric contraction semigroups.

Since, by (1.1)

$$m(\mathbb{L}_\alpha)\omega = \sum_I m(\mathcal{L}_{\alpha,I})\omega_I dx_I,$$

via a randomisation argument based on Rademacher's function we may reduce the problem to studying spectral multipliers of the operators $\mathcal{L}_{\alpha,I}$. When $\alpha \in [-1/2, \infty)^d$ the operators $\mathcal{L}_{\alpha,I}$, generate contractions semigroups on $L^p(\mathbb{R}_+^d, \mu_\alpha)$, $1 \leq p < \infty$. Therefore Carbonaro and Dragičević's result applies to them. Exploiting these facts, we prove that if m is a bounded holomorphic function in the translated sector $S_{\phi_p^*} + r/2$, satisfying suitable Hörmander type conditions on the boundary, then the operator $m(\mathbb{L}_\alpha)$ is bounded on $L^p(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$. We emphasise the fact that our estimates depend on the dimension $\binom{d}{r}$ of the space of alternating tensors of rank r on \mathbb{R}^d .

Now we describe in some detail how the paper is organised. Section 2 contains the results on Laguerre operators on functions. In Section 2.1 we describe the setup and we introduce the operator \mathcal{L}_α , the Laguerre derivatives δ_i , $i = 1, \dots, d$ and their adjoints δ_i^* . We recall the spectral resolution of \mathcal{L}_α and the definition of the Laguerre semigroup. In Section 2.2 we define the generalised Laguerre operators $\mathcal{L}_{\alpha,I}$, we give their spectral resolutions and we describe the properties of the heat and Poisson semigroups generated by them.

In Section 3 we define the Hodge-Laguerre operator \mathbb{L}_α on forms and prove its basic properties. After a few preliminaries on differential forms, in Section 3.2 we define first δ , δ^* and $\mathbb{L}_\alpha = \delta\delta^* + \delta^*\delta$ on smooth forms. Then, in Section 3.3 we prove that \mathbb{L}_α acts diagonally on the coefficients of the form, i.e. formula (1.1). In Section 3.4 we tackle the problem of defining δ , δ^* and \mathbb{L}_α as closed densely defined operators on $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$. Since for $\alpha \notin [1, \infty)^d$ the operator \mathbb{L}_α has several self-adjoint extensions, we must specify the specific extension we work with. This is done by choosing an appropriate orthonormal basis B_r of $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ consisting of eigenfunctions of \mathbb{L}_α . Then the domains of the operators δ , δ^* and \mathbb{L}_α are described in terms of size conditions on the coefficients of a form with respect to the orthonormal basis B_r . The map that sends a form $\omega \in L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ to the multisequence of its coefficients with respect to the basis B_r can be seen as a map from $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ to a space of square summable multi-sequences of alternating covariant tensors of rank r on \mathbb{R}^d , that we call *Fourier-Laguerre* transform. Then we derive simple and elegant formulas relating the Fourier-Laguerre transform of a form ω and those of the forms $\delta\omega$, $\delta^*\omega$ and $\mathbb{L}_\alpha\omega$. These formulas are useful in proving the fundamental identities $\delta^2 = 0$, $(\delta^*)^2 = 0$ and $\mathbb{L}_\alpha = \delta\delta^* + \delta^*\delta$ on the appropriate domains of these operators. We end this rather long section by proving that for all $\rho \leq r/2$ the “heat” and Poisson semigroups generated by the operators $\mathbb{L}_\alpha - \rho I$ and $(\mathbb{L}_\alpha - \rho I)^{1/2}$ are bounded on $L^p(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ for all $p \in [1, \infty]$. In Section 4 we state the Hodge-de Rham decomposition for the space $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$, whose proof can be obtained using the results of the previous section along classical lines. Then we show that

the Riesz- Laguerre transforms $\mathcal{R} = \delta \mathbb{L}_\alpha^{-1/2}$ and $\mathcal{R}^* = \mathbb{L}_\alpha^{-1/2} \delta^*$ are bounded on $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ and that the operators $\mathcal{R}\mathcal{R}^*$ and $\mathcal{R}^*\mathcal{R}$ are the Hodge-de Rham projections onto the spaces of exact and coexact forms respectively.

In Section 5 we state the bilinear embedding theorem. Deferring its proof, we deduce from it the boundedness on $L^p(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ of the Riesz-Laguerre transforms. As applications, we deduce the strong Hodge-de Rham decomposition in $L^p(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ and the existence and regularity result for the Hodge system and the de Rham equation in $L^p(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$.

In Section 6, to prepare the proof of the bilinear embedding theorem, we recall the definition and the basic properties of the particular Bellman function used by A. Carbonaro and O. Dragičević in [4] to prove the boundedness of Riesz transforms on Riemannian manifolds. Even though the results coincide with those in [4] we have included full proofs for completeness.

In Section 7 we prove the bilinear embedding theorem, adapting to our situation the arguments in [4].

Finally in Section 8 we state and prove the spectral multiplier theorem for \mathbb{L}_α when $\alpha \in [-1/2, \infty)^d$.

2. LAGUERRE OPERATORS ON FUNCTIONS

2.1. The operator \mathcal{L}_α . Let \mathbb{R}_+^d be the cone $\{x \in \mathbb{R}^d : x_i > 0, \forall i = 1, \dots, d\}$. Given a multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha \in (-1, \infty)^d$, we define the *Laguerre polynomial of type α and multidegree k on \mathbb{R}_+^d* as

$$L_k^\alpha(x) = L_{k_1}^{\alpha_1}(x_1) \cdots L_{k_d}^{\alpha_d}(x_d),$$

where $k = (k_1, \dots, k_d)$ has integer components $k_i \geq 0$ for each $i = 1, \dots, d$, and

$$L_{k_i}^{\alpha_i}(x_i) = \frac{1}{k_i!} e^{x_i} x_i^{-\alpha_i} \frac{d^{k_i}}{dx_i^{k_i}} (e^{-x_i} x_i^{k_i + \alpha_i})$$

is the one-dimensional Laguerre polynomial of type α_i and degree k_i .

These polynomials are orthogonal with respect to the probability measure

$$(2.1) \quad d\mu_\alpha(x) = \rho_\alpha(x) dx = \prod_{i=1}^d \frac{x_i^{\alpha_i} e^{-x_i}}{\Gamma(\alpha_i + 1)} dx_i,$$

which is called the *Laguerre measure on \mathbb{R}_+^d* . We denote by

$$\ell_k^\alpha = L_k^\alpha / \|L_k^\alpha\|_2$$

their normalizations in $L^2(\mathbb{R}_+^d, \mu_\alpha)$. The system $\{\ell_k^\alpha : k \in \mathbb{N}^d\}$ is an orthonormal basis of $L^2(\mathbb{R}_+^d, \mu_\alpha)$ of eigenfunctions of the Laguerre operator of type α

$$\mathcal{L}_\alpha = - \sum_{i=1}^d \left[x_i \frac{\partial^2}{\partial x_i^2} + (\alpha_i + 1 - x_i) \frac{\partial}{\partial x_i} \right].$$

Namely

$$\mathcal{L}_\alpha \ell_k^\alpha = |k| \ell_k^\alpha, \quad \forall k \in \mathbb{N}^d,$$

where $|k| = k_1 + \dots + k_d$ denotes as usual the length of the multiindex k .

The operator \mathcal{L}^α is nonnegative and symmetric on the domain $C_c^\infty(\mathbb{R}_+^d)$ with respect to the inner product in $L^2(\mathbb{R}_+^d, \mu_\alpha)$. If $\alpha_i \geq 1$ for all $i = 1, \dots, d$ then

it is also essentially self-adjoint on $L^2(\mathbb{R}_+^d, \mu_\alpha)$ and the spectral resolution of its closure, denoted by \mathcal{L}_α as well, is

$$(2.2) \quad \mathcal{L}_\alpha = \sum_{n=0}^{\infty} n \mathcal{P}_n^\alpha,$$

where \mathcal{P}_n^α denotes the orthogonal projection on the space spanned by Laguerre polynomials of degree n . If at least one of the indices α_i is such that $\alpha_i < 1$ the operator \mathcal{L}^α , with domain $C_c^\infty(\mathbb{R}_+^d)$, has several selfadjoint realizations, depending on the boundary conditions at $x_i = 0$ [40].

In the following we shall always work with the realization provided by the spectral resolution (2.2) for all $\alpha \in (-1, \infty)^d$. This selfadjoint realisation can also be characterised as the closure of the operator \mathcal{L}^α on the domain $\mathcal{P}(\mathbb{R}_+^d)$ of all finite linear combinations of Laguerre polynomials, i.e. of all polynomial functions on \mathbb{R}_+^d .

The Laguerre operator can be expressed in the more compact way

$$\mathcal{L}^\alpha = \sum_{i=1}^d \delta_i^* \delta_i,$$

which makes the symmetry evident, by introducing the *Laguerre partial derivatives*

$$\delta_i = \sqrt{x_i} \partial_{x_i}$$

and their formal adjoints in $L^2(\mathbb{R}_+^d, \mu_\alpha)$

$$\delta_i^* = -\sqrt{x_i} \left(\partial_{x_i} + \frac{\alpha_i + \frac{1}{2} - x_i}{x_i} \right) = -(\delta_i + \psi_i(x_i)),$$

where

$$\psi_i(x_i) = \delta_i \log \left(x_i^{\alpha_i + \frac{1}{2}} e^{-x_i} \right) = \frac{\alpha_i + \frac{1}{2}}{\sqrt{x_i}} - \sqrt{x_i}.$$

The operator \mathcal{L}_α is the infinitesimal generator of the Laguerre semigroup $T_t^\alpha = e^{-t\mathcal{L}_\alpha}$, $t \geq 0$ on $L^2(\mathbb{R}_+^d, \mu_\alpha)$. Since the set $\{\ell_k^\alpha\}_{k \in \mathbb{N}^d}$ is an orthonormal basis for $L^2(\mathbb{R}_+^d, \mu_\alpha)$, each function f in this space can be expressed as

$$f = \sum_{k \in \mathbb{N}^d} \hat{f}(k) \ell_k^\alpha,$$

where

$$\hat{f}(k) = \langle f, \ell_k^\alpha \rangle_\alpha$$

and $\langle \cdot, \cdot \rangle_\alpha$ denotes the inner product in $L^2(\mathbb{R}_+^d, \mu_\alpha)$. Hence

$$(2.3) \quad T^\alpha f = \sum_{k \in \mathbb{N}^d} e^{-t|k|} \hat{f}(k) \ell_k^\alpha.$$

The Laguerre semigroup acts as a semigroup of integral operators

$$T^\alpha f(x) = \int_{\mathbb{R}_+^d} G_t^\alpha(x, y) f(y) d\mu_\alpha(y).$$

where

$$G_t^\alpha(x, y) = \sum_{k \in \mathbb{N}^d} e^{-t|k|} \ell_k^\alpha(x) \ell_k^\alpha(y)$$

is the *Laguerre heat kernel*, which has the following explicit expression in terms of modified Bessel functions of the first kind I_{α_i}

$$G_t^\alpha(x, y) = \prod_{i=1}^d \Gamma(\alpha_i + 1) (1 - u)^{-1} \exp\left(-\frac{u(x_i + y_i)}{1 - u}\right) \\ \times (\sqrt{x_i y_i u})^{-\alpha_i} I_{\alpha_i}\left(2\frac{\sqrt{x_i y_i u}}{1 - u}\right)$$

with $u = e^{-t}$ [30]. From this expression it is easily seen that $\{T_t^\alpha\}$ is a symmetric diffusion semigroup, i.e. for all $t \geq 0$

- (i) T_t^α is positivity preserving for all $t \geq 0$;
- (ii) $T_t^\alpha 1 = 1$;
- (iii) $\|T_t^\alpha f\|_p \leq \|f\|_p$, $1 \leq p \leq \infty$.

Note that the heat kernel on \mathbb{R}_+^d is the product of the d one-dimensional heat kernels relative to the operators $\mathcal{L}^{\alpha_i} = \delta_i^* \delta_i$ acting on $L^2(\mathbb{R}_+, \mu_{\alpha_i})$

$$(2.4) \quad G_t^\alpha(x, y) = \prod_{i=1}^d G_t^{\alpha_i}(x_i, y_i).$$

2.2. The operators $\mathcal{L}_{\alpha, I}$. In this subsection we define some generalizations of the Laguerre operator \mathcal{L}_α that will play an important role in the analysis of the Hodge-Laguerre operator on forms.

Definition 2.1. Given a subset $I \subseteq \{1, 2, \dots, d\}$ we define the differential operator

$$\mathcal{L}_{\alpha, I} = \sum_{i \notin I} \delta_i^* \delta_i + \sum_{i \in I} \delta_i \delta_i^*.$$

Remark. When $I = \{i\}$ is a singleton, the operators $\mathcal{L}_{\alpha, i}$, $i = 1, \dots, d$, coincide with the operators M_i^α introduced by Nowak in [30] and studied by Nowak and Stempak in [31] in connection with conjugate Poisson integrals.

We observe that $\mathcal{L}_\alpha = \mathcal{L}_{\alpha, \emptyset}$. Moreover, since

$$[\delta_j^*, \delta_j]f(x) = \delta_j \psi_j(x_j) f(x),$$

we have that

$$(2.5) \quad \mathcal{L}_\alpha = \mathcal{L}_{\alpha, I} - M_{\alpha, I}$$

where

$$(2.6) \quad M_{\alpha, I} f(x) = - \left(\sum_{j \in I} \delta_j \psi_j(x_j) \right) f(x).$$

Notice that if $\alpha \in [-1/2, \infty)^d$ then

$$(2.7) \quad M_{\alpha, I} f(x) \geq \frac{\#I}{2} f(x) \quad \forall x \in \mathbb{R}_+^d,$$

where $\#I$ denotes the cardinality of the set I , since

$$-\delta_j \psi_j(x) = \frac{1}{2} \left(\frac{\alpha_j + \frac{1}{2}}{x_j} + 1 \right) \geq 1/2 \quad \forall j = 1, 2, \dots, d.$$

For each $I \subseteq \{1, 2, \dots, d\}$ we denote by $\mathcal{K}(I)$ the set of all multi- indexes $k = (k_1, \dots, k_d)$ in \mathbb{N}^d such that $k_i \geq 1$ for $i \in I$. Note that if $k \in \mathcal{K}(I)$ then $|k| \geq \#I$. For each $I \subseteq \{1, 2, \dots, d\}$ and $k \in \mathcal{K}(I)$ define

$$\ell_k^{\alpha, I}(x_1, \dots, x_d) = \prod_{i \notin I} \ell_{k_i}^{\alpha_i}(x_i) \prod_{i \in I} \sqrt{\frac{\alpha_i + 1}{k}} \delta_i \ell_{k_i}^{\alpha_i}(x_i).$$

Proposition 2.2. *The family of functions*

$$B_I = \left\{ \ell_k^{\alpha, I} : k \in \mathcal{K}(I) \right\}$$

is an orthonormal basis of $L^2(\mathbb{R}_+^d, \mu_\alpha)$ of eigenfunctions of the operator $\mathcal{L}_{\alpha, I}$. Namely

$$\mathcal{L}_{\alpha, I} \ell_k^{\alpha, I} = |k| \ell_k^{\alpha, I}.$$

Proof. In view of the tensor product structure of the functions $\ell_k^{\alpha, I}$, it is sufficient to prove that for $\alpha > -1$ the families $\{\ell_k^\alpha : k \in \mathbb{N}\}$ and $\{\sqrt{\frac{\alpha+1}{k}} \delta \ell_k^\alpha : k \in \mathbb{N}_+\}$ are orthonormal bases of $L^2(\mathbb{R}_+, \mu_\alpha)$. We have already observed in the previous section that the former family is an orthonormal basis. Hence we only need to show that $\{\sqrt{\frac{\alpha+1}{k}} \delta \ell_k^\alpha : k \in \mathbb{N}_+\}$ is an orthonormal basis. Since, by a well known property of Laguerre polynomials, $\partial_t L_k^\alpha(t) = -L_{k-1}^{\alpha+1}(t)$ and $\|L_k^\alpha\|_2 = (\Gamma(\alpha + k + 1)/\Gamma(\alpha + 1)\Gamma(k + 1))^{1/2}$, it is easily seen that

$$\sqrt{\frac{\alpha+1}{k}} \delta \ell_k^\alpha(t) = \sqrt{\frac{\alpha+1}{k}} \sqrt{t} \partial_t \ell_k^\alpha(t) = -\sqrt{t} \ell_{k-1}^{\alpha+1}(t)$$

Since $\{\ell_j^{\alpha+1} : j \in \mathbb{N}\}$ is an orthonormal basis of $L^2(\mathbb{R}_+, \mu_{\alpha+1})$, the desired conclusion follows immediately.

The fact that $\ell_k^{\alpha, I}$ is an eigenfunction of $\mathcal{L}_{\alpha, I}$ with eigenvalue $|k|$ follows from the identities $\delta_j^* \delta_j \ell_{k_j}^{\alpha_j} = k_j \ell_{k_j}^{\alpha_j}$ and $\delta_j \delta_j^* \delta_j \ell_{k_j}^{\alpha_j} = k_j \delta_j \ell_{k_j}^{\alpha_j}$. \square

With a sligh abuse of notation we shall denote by $\mathcal{L}_{\alpha, I}$ also the self-adjoint extension of $\mathcal{L}_{\alpha, I}$ with spectral resolution

$$\mathcal{L}_{\alpha, I} = \sum_{n \geq \#I} n \mathcal{P}_{I, n},$$

where $\mathcal{P}_{I, n}$ is the orthogonal projection onto the subspace spanned by the functions $\ell_k^{\alpha, I}$, $k \in \mathcal{K}(I)$, $|k| = n$.

We denote by $\{T_t^{\alpha, I} : t \geq 0\}$ the semigroup generated by $-\mathcal{L}_{\alpha, I}$ and by $\{P_t^{\alpha, I} : t \geq 0\}$ the corresponding Poisson semigroup, generated by $-(\mathcal{L}_{\alpha, I})^{1/2}$. When $I = \emptyset$ we shall simply write T_t^α and P_t^α instead of $T_t^{\alpha, \emptyset}$ and $P_t^{\alpha, \emptyset}$. These semigroups have the spectral representations

$$T_t^{\alpha, I} f = \sum_{k \in \mathcal{K}(I)} e^{-t|k|} \hat{f}(I, k) \ell_k^{\alpha, I},$$

and

$$P_t^{\alpha, I} f = \sum_{k \in \mathcal{K}(I)} e^{-t|k|^{1/2}} \hat{f}(I, k) \ell_k^{\alpha, I},$$

where

$$\hat{f}(I, k) = \langle f, \ell_k^{\alpha, I} \rangle_\alpha.$$

The Poisson semigroup can also be defined via the subordination principle

$$P_t^{\alpha, I} f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} T_{t^2/4u}^{\alpha, I} f(x) du.$$

The semigroup $\{T_t^{\alpha, I} : t \geq 0\}$ has also the integral representation

$$T_t^{\alpha, I} f(x) = \int_{\mathbb{R}_+^d} G_t^{\alpha, I}(x, y) f(y) d\mu_\alpha(y),$$

where

$$(2.8) \quad G_t^{\alpha, I}(x, y) = \prod_{i \notin I} G_t^{\alpha_i}(x_i, y_i) \prod_{i \in I} \tilde{G}_t^{\alpha_i}(x_i, y_i)$$

and

$$\tilde{G}_t^{\alpha_i}(x_i, y_i) = e^{-t} \sqrt{x_i y_i} G_t^{\alpha_i+1}(x_i, y_i)$$

is the kernel of the semigroup generated by the operator $\delta_j \delta_j^*$ on $L^2(\mathbb{R}_+, \mu_{\alpha_j})$ (see [30]). The following lemma has been proved by A. Nowak.

Lemma 2.3. *There exists a non increasing function Λ on $(-1, \infty)$ such that $\Lambda(\nu) = 1$ for $\nu \geq -1/2$, $\Lambda(\nu) = O((\nu + 1)^{-1/2})$ for $\nu \rightarrow -1$ and*

$$\tilde{G}_t^\nu(x, y) \leq \Lambda(\nu) e^{-t/2} G_t^\nu(x, y) \quad \forall x, y \in \mathbb{R}_+, t > 0.$$

Proof. See [30, Lemma 2]. □

Proposition 2.4. *For every $\alpha \in (-1, \infty)^d$ there exists a constant $C(\alpha)$ such that $C(\alpha) = 1$ if $\alpha_i \geq -1/2$ for all $i \in I$ and*

- (i) $|T_t^{\alpha, I} f(x)| \leq C(\alpha) e^{-\#I t/2} T_t^\alpha |f|(x),$
- (ii) $|P_t^{\alpha, I} f(x)| \leq C(\alpha) P_t^\alpha |f|(x),$

for all $f \in L^2(\mathbb{R}_+^d, \mu_\alpha)$, $x \in \mathbb{R}^d$, $t > 0$.

Proof. Set $C(\alpha) = \prod_{i \in I} \Lambda(\alpha_i)$. Then by (2.8) and Lemma 2.3,

$$G_t^{\alpha, I}(x, y) \leq C(\alpha) e^{-\#I t/2} G_t^\alpha(x, y).$$

Thus (i) follows by the positivity of the kernel $G_t^{\alpha, I}(x, y)$ and (ii) follows from the subordination principle. □

Corollary 2.5. *For every $\alpha \in (-1, \infty)^d$ there exists a constant $C(\alpha)$ such that $C(\alpha) = 1$ if $\alpha_i \geq -1/2$ for all $i \in I$ and*

$$\|T_t^{\alpha, I}\|_{p-p} \leq C(\alpha)^{(2/p)-1} \exp \left[-t \#I \left(1 - \left| \frac{1}{2} - \frac{1}{p} \right| \right) \right] \quad \forall p \in [1, \infty].$$

In particular, if $\alpha_i \geq -1/2$ for all $i \in I$, then $\mathcal{L}_{\alpha, I} - (\#I/2)I$ generates a symmetric semigroup of contractions on $L^p(\mathbb{R}_+^d, \mu_\alpha)$ for every $p \in [1, \infty]$.

Proof. The estimate of $\|T_t^{\alpha, I}\|_{1-1}$ follows from Proposition 2.4, since T_t^α is a contraction on $L^1(\mathbb{R}_+^d, \mu_\alpha)$. The estimate of $\|T_t^{\alpha, I}\|_{2-2}$ follows from the spectral resolution of $\mathcal{L}_{\alpha, I}$. The general case follows by interpolation and duality. □

3. THE HODGE-LAGUERRE OPERATOR

In this section we define the Hodge-Laguerre operator on differential forms on \mathbb{R}_+^d and prove his basic properties. In the first two subsections we recall briefly the definition of differential forms, and the basic algebraic operations on them that we shall need in the sequel: the exterior and the interior products, the Hodge-star operator and their properties. These results are classical and we refer the reader to the monograph [38] of F. W. Warner for complete proofs. The main purpose of this preliminary section is to establish notation and terminology. In the next subsection 3.2 we define the Laguerre exterior differential δ and its formal adjoint δ^* with respect to the Laguerre measure μ_α .

3.1. Differential forms on \mathbb{R}_+^d . For each $r \in \{0, 1, \dots, d\}$ we denote by $\Lambda^r = \Lambda^r(\mathbb{R}^d)$ the space of real alternating tensors of rank r on \mathbb{R}^d . For every r we denote by \mathcal{I}_r the set of all multiindices (i_1, i_2, \dots, i_r) such that $i_1 < i_2 < \dots < i_r$. The space Λ^r is endowed with the inner product $\langle \omega, \eta \rangle_{\Lambda^r}$ and the corresponding norm $|\omega|_{\Lambda^r}$ for which the set of covectors

$$dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_r}, \quad I \in \mathcal{I}_r$$

is an orthonormal basis. Often we shall simply denote by $\langle \omega, \eta \rangle$ and $|\omega|$ the inner product and the norm, omitting the subscript Λ^r when there is no risk of confusion. Thus if $\omega = \sum_{I \in \mathcal{I}_r} \omega_I dx_I$ and $\eta = \sum_{I \in \mathcal{I}_r} \eta_I dx_I$ are two elements of Λ^r their inner product is $\langle \omega, \eta \rangle = \sum_{I \in \mathcal{I}_r} \omega_I \eta_I$. We shall denote by $*$ the Hodge-star isomorphism of the exterior algebra, mapping Λ^r to Λ^{d-r} for each r . Then, if we denote by \mathbb{I} the volume form $dx_1 \wedge \dots \wedge dx_d$,

$$\omega \wedge * \eta = \langle \omega, \eta \rangle \mathbb{I} \quad \forall \omega, \eta \in \Lambda^r.$$

If $\omega \in \Lambda^r$ we denote by $\iota_\omega : \Lambda^{s+r} \rightarrow \Lambda^s$ the operator of interior multiplication by ω , i.e. the adjoint of exterior multiplication by ω with respect to the inner product on Λ^r .

Lemma 3.1. *If $\phi \in \Lambda^1(\mathbb{R}^d)$ and $\omega \in \Lambda^r(\mathbb{R}^d)$, $0 \leq r \leq d$, then*

$$(3.1) \quad \phi \wedge \iota_\phi(\omega) + \iota_\phi(\phi \wedge \omega) = |\phi|^2 \omega$$

$$(3.2) \quad |\phi \wedge \omega|^2 + |\iota_\phi \omega|^2 = |\phi|^2 |\omega|^2.$$

Proof. If $I = (i_1, \dots, i_r) \in \mathcal{I}_r$ and $1 \leq j \leq d$, we denote by $\sigma(j, I)$ the number of components of I which are strictly less than j .

If $j \notin I$ we denote by $I \cup j$ the element of \mathcal{I}_{r+1} obtained by adding j to the components of I ; if $j \in I$ we denote by $I \setminus j$ the element of \mathcal{I}_{r-1} obtained by deleting j from I . Thus

$$\begin{aligned} dx_j \wedge dx_I &= (-1)^{\sigma(j, I)} dx_{I \cup j} \\ \iota_{dx_j}(dx_I) &= (-1)^{\sigma(j, I)} dx_{I \setminus j}. \end{aligned}$$

Let $\phi = \sum_{j=1}^d \phi_j dx_j$ and $\omega = \sum_{I \in \mathcal{I}_r} \omega_I dx_I$. Then, on the one hand,

$$\begin{aligned}
\phi \wedge \iota_\phi(\omega) &= \phi \wedge \sum_{I \in \mathcal{I}_r} \sum_{j \in I} \phi_j \omega_I \iota_{dx_j}(dx_I) \\
&= \phi \wedge \sum_{I \in \mathcal{I}_r} \sum_{j \in I} (-1)^{\sigma(j, I)} \phi_j \omega_I dx_{I \setminus j} \\
&= \sum_{I \in \mathcal{I}_r} \sum_{j \in I} \sum_{i \notin \{I \setminus j\}} (-1)^{\sigma(j, I)} \phi_i \phi_j \omega_I dx_i \wedge dx_{I \setminus j} \\
&= \sum_{I \in \mathcal{I}_r} \sum_{j \in I} \sum_{i \notin \{I \setminus j\}} (-1)^{\sigma(j, I) + \sigma(i, I \setminus j)} \phi_i \phi_j \omega_I dx_{(I \setminus j) \cup i}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\iota_\phi(\phi \wedge \omega) &= \iota_\phi \left(\sum_{I \in \mathcal{I}_r} \sum_{i \notin I} \phi_i \omega_I dx_i \wedge dx_I \right) \\
&= \iota_\phi \left(\sum_{I \in \mathcal{I}_r} \sum_{i \notin I} (-1)^{\sigma(i, I)} \phi_i \omega_I dx_{I \cup i} \right) \\
&= \sum_{I \in \mathcal{I}_r} \sum_{i \notin I} \sum_{j \in \{I \cup i\}} (-1)^{\sigma(i, I)} \phi_i \phi_j \omega_I \iota_{dx_j}(dx_{I \cup i}) \\
&= \sum_{I \in \mathcal{I}_r} \sum_{i \notin I} \sum_{j \in \{I \cup i\}} (-1)^{\sigma(i, I) + \sigma(j, I \cup i)} \phi_j \phi_i \omega_I dx_{(I \cup i) \setminus j}.
\end{aligned}$$

Next we observe that in the sum $\phi \wedge \iota_\phi(\omega) + \iota_\phi(\phi \wedge \omega)$ the terms containing indices $i \neq j$ cancel out. Indeed,

- (a) $\{(i, j) : i \neq j, j \in I, i \notin I \setminus j\} = \{(i, j) : i \neq j, i \notin I, j \in I \cup i\}$
- (b) if $i \neq j$ the exponents $\sigma(j, I) + \sigma(i, I \setminus j)$ and $\sigma(i, I) + \sigma(j, I \cup i)$ have opposite parity, as it can be easily seen by observing that if $i < j$ then

$$\sigma(i, I \setminus j) = \sigma(i, I) \quad \text{and} \quad \sigma(j, I \cup i) = \sigma(j, I) + 1,$$

while, if $j < i$ then

$$\sigma(j, I \cup i) = \sigma(j, I) \quad \text{and} \quad \sigma(i, I) = \sigma(i, I \setminus j) + 1.$$

Therefore, only the terms with $i = j$ remain and, since for $i = j$ the exponents of -1 are even, we have that

$$\begin{aligned}
\phi \wedge \iota_\phi(\omega) + \iota_\phi(\phi \wedge \omega) &= \sum_{I \in \mathcal{I}_r} \sum_{i \in I} \phi_i \phi_i \omega_I dx_I + \sum_{I \in \mathcal{I}_r} \sum_{i \notin I} \phi_i \phi_i \omega_I dx_I \\
&= |\phi|^2 \omega.
\end{aligned}$$

This proves (3.1). To prove (3.2) observe that

$$\begin{aligned}
|\phi \wedge \omega|^2 + |\iota_\phi(\phi \wedge \omega)|^2 &= \langle \phi \wedge \omega, \phi \wedge \omega \rangle + \langle \iota_\phi \omega, \iota_\phi \omega \rangle \\
&= \langle \omega, \iota_\phi(\phi \wedge \omega) \rangle + \langle \omega, \phi \wedge \iota_\phi \omega \rangle \\
&= |\phi|^2 |\omega|^2.
\end{aligned}$$

□

We shall denote by $C^\infty(\mathbb{R}_+^d; \Lambda^r)$ the space of differential forms of order r on \mathbb{R}_+^d with smooth coefficients and by $C_c^\infty(\mathbb{R}_+^d; \Lambda^r)$ those with compact support. For every $p \in [1, \infty]$ we denote by $L^p(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ the space of forms of order r with coefficients in $L^p(\mathbb{R}_+^d, \mu_\alpha)$, endowed with the norm

$$\|\omega\|_p = \left(\int_{\mathbb{R}_+^d} |\omega(x)|^p d\mu_\alpha(x) \right)^{1/p}$$

with the usual modification when $p = \infty$. If ω, η are in $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$, we denote by

$$\langle \omega, \eta \rangle_\alpha = \int_{\mathbb{R}_+^d} \langle \omega(x), \eta(x) \rangle_{\Lambda^r} d\mu_\alpha(x)$$

their inner product in $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$. To simplify notation, sometimes we shall write simply $L^p(\mu_\alpha; \Lambda^r)$ or $L^p(\Lambda^r)$ instead of $L^p(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$.

3.2. The Hodge-Laguerre operator on forms. In this subsection we define the Hodge-Laguerre operator acting on smooth forms in \mathbb{R}_+^d as a natural generalisation of the Laguerre operator on functions. We begin by defining the Laguerre exterior derivative operator δ and its formal adjoint, the Laguerre codifferential δ^* .

If $\omega = \sum_{I \in \mathcal{I}_r} \omega_I dx_I$ is an r -form in $C^\infty(\mathbb{R}_+^d; \Lambda^r)$, its Laguerre exterior differential is the $(r+1)$ -form

$$\delta\omega = \sum_{j=1}^d \sum_{I \in \mathcal{I}_r} \delta_j \omega_I dx_j \wedge dx_I,$$

where δ_j denotes the differential operator $\sqrt{x_j} \partial_j$. Using the trivial fact that the partial derivatives δ_i and δ_j commute for $i \neq j$, it is easy to see that $\delta^2 = 0$. Furthermore δ is an antiderivation, i.e.

$$\delta(\omega \wedge \eta) = \delta\omega \wedge \eta + (-1)^r \omega \wedge \delta\eta$$

for all r -forms ω and s -forms η .

The Laguerre codifferential δ^* is the formal adjoint of δ with respect to the inner product in $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$. In other words, if ω is a form in $C^\infty(\Lambda^{r+1}(\mathbb{R}_+^d))$ then $\delta^*\omega$ is the r -form defined by the identity

$$\langle \delta^*\omega, \eta \rangle_\alpha = \langle \omega, \delta\eta \rangle_\alpha \quad \forall \eta \in C_c^\infty(\mathbb{R}_+^d; \Lambda^r).$$

We define δ^* also on 0-forms by setting $\delta^*\omega = 0$ for each smooth 0-form ω .

It follows immediately from the definition of δ^* that $(\delta^*)^2 = 0$. We give two more explicit expressions of δ^* . To this purpose we introduce the 1-form

$$\psi(x) = \sum_{j=1}^d \psi_j(x_j) dx_j,$$

where

$$\psi_j(x_j) = \frac{\alpha_j + 1/2}{\sqrt{x_j}} - \sqrt{x_j} = \frac{1}{\rho_\alpha(x)} \partial_j (\sqrt{x_j} \rho_\alpha(x)),$$

where ρ_α denotes the Laguerre density (see (2.1)). The following two propositions give two representations of the action of δ^* on r -forms.

Proposition 3.2. On $C_c^\infty(\mathbb{R}_+^d; \Lambda^r)$

$$\delta^* = (-1)^{d(r-1)+1} *_{d-r+1} \delta *_{r-1} \iota_\psi,$$

where $*_r$ and $*_{d-r+1}$ denote the restrictions of the Hodge $*$ -operator to r -forms and to $(d-r+1)$ -forms, respectively, and ι_ψ is the interior multiplication by the form ψ .

Proposition 3.3. If $\omega \in C_c^\infty(\mathbb{R}_+^d; \Lambda^r)$ then

$$\delta^* \omega = \sum_{I \in \mathcal{I}_r} \sum_{j=1}^d \delta_j^* \omega_I \iota_{dx_j}(dx_I),$$

where $\delta_j^* = -(\sqrt{x_j} \partial_j + \psi_j(x_j))$.

To prove Proposition 3.2, it is convenient to state a lemma.

Lemma 3.4. Denote by \mathbb{I} the volume element $dx_1 \wedge \dots \wedge dx_d$ on \mathbb{R}_+^d . Then

$$\delta^* \mathbb{I} = - *_{\mathbf{1}} \psi.$$

Proof. Let $\omega = \sum_{i=1}^d \omega_i \widehat{dx_i}$ be a form in $C_c^\infty(\mathbb{R}_+^d; \Lambda^{d-1})$, where

$$\widehat{dx_i} = (-1)^{i-1} *_{\mathbf{1}} dx_i = dx_1 \wedge \dots \wedge \underline{dx_i} \wedge \dots \wedge dx_d$$

with the element dx_i omitted. Since

$$dx_j \wedge \widehat{dx_i} = \begin{cases} 0 & \text{if } j \neq i \\ (-1)^{i-1} \mathbb{I} & \text{if } j = i, \end{cases}$$

we have that

$$\delta \omega = \sum_{i=1}^d \sum_{j=1}^d \delta_j \omega_i dx_j \wedge \widehat{dx_i} = \sum_{i=1}^d (-1)^{i-1} \delta_i \omega_i \mathbb{I}.$$

Hence

$$\begin{aligned} \langle \delta^* \mathbb{I}, \omega \rangle_\alpha &= \langle \mathbb{I}, \delta \omega \rangle_\alpha \\ &= \sum_{i=1}^d (-1)^{i-1} \int_{\mathbb{R}_+^d} \delta_i \omega_i d\mu_\alpha \\ &= - \sum_{i=1}^d (-1)^{i-1} \int_{\mathbb{R}_+^d} \omega_i \partial_i (\sqrt{x_i} \rho_\alpha(x)) dx \\ &= - \sum_{i=1}^d (-1)^{i-1} \int_{\mathbb{R}_+^d} \omega_i \psi_i d\mu_\alpha = - \langle *_{\mathbf{1}} \psi, \omega \rangle_\alpha. \end{aligned}$$

Since the choice of ω is arbitrary, it follows that

$$\delta^* \mathbb{I} = - *_{\mathbf{1}} \psi.$$

□

Proof of Proposition 3.2. Suppose that $\omega \in C_c^\infty(\mathbb{R}_+^d; \Lambda^{r-1})$ and $\eta \in C^\infty(\mathbb{R}_+^d; \Lambda^r)$. Then

$$(3.3) \quad \langle \omega, \delta^* \eta \rangle_\alpha = \langle \delta \omega, \eta \rangle_\alpha = \int_{\mathbb{R}_+^d} \delta \omega \wedge *_{\mathbf{r}} \eta \rho_\alpha.$$

By the antiderivation property of δ

$$\delta(\omega \wedge *_r \eta) = (\delta\omega) \wedge *_r \eta + (-1)^{r-1} \omega \wedge \delta(*_r \eta).$$

Thus

$$(3.4) \quad \int_{\mathbb{R}_+^d} \delta\omega \wedge *_r \eta \, \rho_\alpha = \int_{\mathbb{R}_+^d} \delta(\omega \wedge *_r \eta) \rho_\alpha - (-1)^{r-1} \int_{\mathbb{R}_+^d} \omega \wedge \delta(*_r \eta) \, \rho_\alpha.$$

We evaluate separately the two integrals. On the one hand

$$\begin{aligned} \int_{\mathbb{R}_+^d} \delta(\omega \wedge *_r \eta) \rho_\alpha &= \langle \delta(\omega \wedge *_r \eta), \mathbb{I} \rangle_\alpha \\ &= \langle \omega \wedge *_r \eta, \delta^* \mathbb{I} \rangle_\alpha \\ &= -\langle \omega \wedge *_r \eta, *_1 \psi \rangle_\alpha \\ &= -\int_{\mathbb{R}_+^d} \omega \wedge *_r \eta \, *_d \psi \, \rho_\alpha \\ &= -(-1)^{d-1} \int_{\mathbb{R}_+^d} \omega \wedge *_r \eta \wedge \psi \, \rho_\alpha \\ &= -(-1)^{2(d-1)} \int_{\mathbb{R}_+^d} \psi \wedge \omega \wedge *_r \eta \, \rho_\alpha \\ &= -\langle \psi \wedge \omega, \eta \rangle_\alpha \\ &= -\langle \omega, \iota_\psi \eta \rangle_\alpha. \end{aligned}$$

Here we have used Lemma 3.4 in the third equality, the fact that $*_{d-1} *_1 = (-1)^{d-1} \text{id}$ in the fifth, and the anticommutativity of the wedge product in the sixth. Thus

$$(3.5) \quad \int_{\mathbb{R}_+^d} \delta(\omega \wedge *_r \eta) \rho_\alpha = -\langle \omega, \iota_\psi \eta \rangle_\alpha.$$

On the other hand, since $\delta *_r \eta = (-1)^{(r-1)(d-r+1)} *_r \delta *_r \eta$,

$$(3.6) \quad (-1)^{r-1} \int_{\mathbb{R}_+^d} \omega \wedge \delta(*_r \eta) \, \rho_\alpha = (-1)^{d(r-1)} \langle \omega, *_r \delta *_r \eta \rangle_\alpha.$$

By combining identities (3.3)-(3.6) we obtain

$$\langle \omega, \delta^* \eta \rangle_\alpha = \langle \omega, ((-1)^{d(r+1)+1} *_r \delta *_r - \iota_\psi) \eta \rangle_\alpha,$$

which is the desired conclusion. \square

Proof of Proposition 3.3. Let $\eta = \sum_J \eta_J dx_J$ be a form in $C^\infty(\mathbb{R}_+^d; \Lambda^{r-1})$. Then

$$\begin{aligned}
 \langle \delta^* \omega, \eta \rangle_\alpha &= \langle \omega, \delta \eta \rangle_\alpha = \sum_I \sum_J \sum_{j=1}^d \langle \omega_I dx_I, \delta_j \eta_J dx_j \wedge dx_J \rangle_\alpha \\
 (3.7) \quad &= \sum_I \sum_J \sum_{j=1}^d \int_{\mathbb{R}_+^d} \omega_I \delta_j \eta_J dx_I \wedge *_{r-1} (dx_j \wedge dx_J) \rho_\alpha \\
 &= \sum_I \sum_J \sum_{j=1}^d \int_{\mathbb{R}_+^d} \omega_I \delta_j \eta_J \iota_{dx_j} (dx_I) \wedge *_{r-1} dx_J \rho_\alpha \\
 &= \sum_I \sum_J \sum_{j=1}^d C(I, J, j) \int_{\mathbb{R}_+^d} \omega_I \delta_j \eta_J d\mu_\alpha.
 \end{aligned}$$

Here we have used the fact that $dx_I \wedge *_{r-1} (dx_j \wedge dx_J) \rho_\alpha = C(I, J, j) d\mu_\alpha$, because $dx_I \wedge *_{r-1} (dx_j \wedge dx_J)$ is a constant multiple of the volume form, with a coefficient $C(I, J, j) \in \{-1, 0, 1\}$. Now, integrating by parts, we obtain that

$$\int_{\mathbb{R}_+^d} \omega_I \delta_j \eta_J d\mu_\alpha = \int_{\mathbb{R}_+^d} \delta_j^* \omega_I \eta_J d\mu_\alpha.$$

Thus, tracing back the chain of identities (3.7) after this integration by parts, we obtain that

$$\langle \delta^* \omega, \eta \rangle_\alpha = \left\langle \sum_I \delta_j^* \omega_I \iota_{dx_j} (dx_I), \eta \right\rangle_\alpha,$$

which is desired conclusion. \square

The *Hodge-Laguerre operator* \mathbb{L}_α is defined by

$$\mathbb{L}_\alpha = \delta \delta^* + \delta^* \delta,$$

and is a linear operator on $C_c^\infty(\mathbb{R}_+^d; \Lambda^r)$ for each r with $0 \leq r \leq d$.

By using the expression of δ^* given in Proposition 3.3 it is easy to check that on 0-forms the Hodge-Laguerre operator \mathbb{L}_α coincides with the Laguerre operator \mathcal{L}_α on functions defined in Section 2.1. Indeed one has

Lemma 3.5. *If $f \in C_c^\infty(\mathbb{R}_+^d; \Lambda^0) = C_c^\infty(\mathbb{R}_+^d)$ then*

$$\mathbb{L}_\alpha f = - \sum_{i=1}^d (\delta_i + \psi_i) \delta_i f.$$

Proof. Since $\delta^* f = 0$, by Proposition 3.3

$$\mathbb{L}_\alpha f = \delta^* \delta f = \sum_{i=1}^d \sum_{j=1}^d \delta_i^* \delta_j f \iota_{dx_i} dx_j = \sum_{i=1}^d \delta_i^* \delta_i f = - \sum_{i=1}^d (\delta_i + \psi_i) \delta_i f.$$

\square

3.3. The diagonalization of the Hodge-Laguerre operator. In this subsection we prove that the action of the Hodge-Laguerre operator on r -forms can be diagonalised with respect to the basis $\{dx_I : I \in \mathcal{I}_r\}$ of Λ^r . Namely

Proposition 3.6. *If $\omega = \sum_{I \in \mathcal{I}_r} \omega_I dx_I \in C^\infty(\mathbb{R}_+^d; \Lambda^r)$ then*

$$\mathbb{L}_\alpha \omega = \sum_{I \in \mathcal{I}_r} \mathcal{L}_{\alpha, I} \omega_I dx_I,$$

where

$$\mathcal{L}_{\alpha, I} = \sum_{j \in I} \delta_j \delta_j^* + \sum_{j \notin I} \delta_j^* \delta_j$$

are the differential operators acting on scalar functions defined in Section 2.2.

Proof. If $\omega \in C^\infty(\mathbb{R}_+^d; \Lambda^r)$, then

$$\mathbb{L}_\alpha \omega = \delta \delta^* \omega + \delta^* \delta \omega.$$

We compute separately the two summands. As before, we denote by $\sigma(j, I)$ the number of components of I which are strictly less than j . On the one hand

$$\begin{aligned} \delta \delta^* \omega &= \delta \sum_{I \in \mathcal{I}_r} \sum_{j \in I} \delta_j^* \omega_I \iota_{dx_j}(dx_I) \\ &= \delta \sum_{I \in \mathcal{I}_r} \sum_{j \in I} (-1)^{\sigma(j, I)} \delta_j^* \omega_I dx_{I \setminus j} \\ &= \sum_{I \in \mathcal{I}_r} \sum_{j \in I} \sum_{i \notin \{I \setminus j\}} (-1)^{\sigma(j, I)} \delta_i \delta_j^* \omega_I dx_i \wedge dx_{I \setminus j} \\ &= \sum_{I \in \mathcal{I}_r} \sum_{j \in I} \sum_{i \notin \{I \setminus j\}} (-1)^{\sigma(j, I) + \sigma(i, I \setminus j)} \delta_i \delta_j^* \omega_I dx_{(I \setminus j) \cup i}. \end{aligned}$$

On the other hand

$$\begin{aligned} \delta^* \delta \omega &= \delta^* \sum_{I \in \mathcal{I}_r} \sum_{i \notin I} \delta_i \omega_I dx_i \wedge dx_I \\ &= \delta^* \sum_{I \in \mathcal{I}_r} \sum_{i \notin I} (-1)^{\sigma(i, I)} \delta_i \omega_I dx_{I \cup i} \\ &= \sum_{I \in \mathcal{I}_r} \sum_{i \notin I} \sum_{j \in \{I \cup i\}} (-1)^{\sigma(i, I)} \delta_i^* \delta_j \omega_I \iota_{dx_j}(dx_{I \cup i}) \\ &= \sum_{I \in \mathcal{I}_r} \sum_{i \notin I} \sum_{j \in \{I \cup i\}} (-1)^{\sigma(i, I) + \sigma(j, I \cup i)} \delta_j^* \delta_i \omega_I dx_{(I \cup i) \setminus j}. \end{aligned}$$

Next we observe that in the sum $\delta^* \delta \omega + \delta \delta^* \omega$ the terms containing indices $i \neq j$ cancel out. Indeed, if $i \neq j$ the differential operators δ_i and δ_i^* commute with δ_j and δ_j^* because they act on different variables. Therefore, only the terms with $i = j$ remain and, since for $i = j$ the exponents of -1 are even, we have that

$$\delta^* \delta \omega + \delta \delta^* \omega = \sum_{I \in \mathcal{I}_r} \sum_{i \in I} \delta_i \delta_i^* \omega_I dx_I + \sum_{I \in \mathcal{I}_r} \sum_{i \notin I} \delta_i^* \delta_i \omega_I dx_I.$$

□

3.4. A self-adjoint extension of the Hodge-Laguerre operator. The operator \mathbb{L}_α with domain $C_c^\infty(\mathbb{R}_+^d; \Lambda^r)$ is obviously symmetric with respect to the inner product $\langle \cdot, \cdot \rangle_\alpha$, but it is not self-adjoint on $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$. In this subsection, to define a self-adjoint extension of \mathbb{L}_α , we modify the domains of the operators δ , δ^* and \mathbb{L}_α . With the new domains the operators δ and δ^* will be adjoint to each other and \mathbb{L}_α will be self-adjoint. The domains will be defined via the Fourier-Laguerre transform of forms, that we define presently. To this end, first we introduce an orthonormal basis for the space of square integrable r -forms.

Proposition 3.7. *The family of r -forms*

$$B_r = \left\{ \ell_k^{\alpha, I} dx_I : I \in \mathcal{I}_r, k \in \mathcal{K}(I) \right\}$$

is an orthonormal basis of $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$.

Proof. By Proposition 2.2 the families $\{\ell_{k_i}^{\alpha_i} : k_i \in \mathbb{N}\}$ and $\{\frac{1}{\sqrt{k_i}} \delta_i \ell_{k_i}^{\alpha_i} : k_i \in \mathbb{N}_+\}$ are orthonormal bases of $L^2(\mathbb{R}_+, \mu_{\alpha_i})$. Thus, by tensorization, $\{\ell_k^{\alpha, I} : k \in \mathcal{K}(I)\}$ is an orthonormal basis of $L^2(\mathbb{R}_+^d, \mu_\alpha)$ for each $I \in \mathcal{I}_r$, and

$$C_r = \left\{ \oplus_{I \in \mathcal{I}_r} (0, \dots, 0, \ell_k^{\alpha, I}, 0, \dots, 0) : k \in \mathcal{K}(I) \right\}$$

is an orthonormal basis of the direct sum $\mathcal{H} = \oplus_{I \in \mathcal{I}_r} L^2(\mathbb{R}_+^d, \mu_\alpha)$ of $\#\mathcal{I}_r$ copies of $L^2(\mathbb{R}_+^d, \mu_\alpha)$. Since the map $\omega \mapsto (\omega_I)_{I \in \mathcal{I}_r}$ is an isometric isomorphism from $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ to \mathcal{H} that maps B_r to C_r , the family B_r is an orthonormal basis of $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$. \square

Definition. The *Fourier-Laguerre coefficients* of a form $\omega \in L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ are the coefficients of ω with respect to the basis B_r , i.e.

$$\hat{\omega}(I, k) = \langle \omega_I, \ell_k^{\alpha, I} \rangle_\alpha, \quad I \in \mathcal{I}_r, k \in \mathcal{K}(I).$$

It is convenient to define $\hat{\omega}(I, k)$ for all $k \in \mathbb{N}^d$, by setting $\hat{\omega}(I, k) = 0$ when $k \notin \mathcal{K}(I)$. Observe that if $I \in \mathcal{I}_r$ then

$$\mathcal{K}(I) \subset \mathbb{N}_r^d = \{k \in \mathbb{N}^d : |k| \geq r\}.$$

To obtain nice formulas for the Fourier-Laguerre transform of the forms $\delta\omega$, $\delta^*\omega$ and $\mathbb{L}_\alpha\omega$ it is useful to give some algebraic structure to the set of Fourier-Laguerre coefficients. Therefore, we define the *Fourier-Laguerre transform* of the form $\omega \in L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ as the multi-sequence of alternating tensors of rank r

$$\mathbb{N}_r^d \ni k \mapsto \hat{\omega}(k) = \sum_{I \in \mathcal{I}_r} \hat{\omega}(I, k) dx_I \in \Lambda^r,$$

and we denote by λ_k^α the r -form defined by

$$\lambda_k^\alpha(x) = \sum_{I \in \mathcal{I}_r} \ell_k^{\alpha, I}(x) dx_I, \quad \forall k \in \mathbb{N}^d.$$

Define the bilinear map $[\cdot, \cdot] : \Lambda^r \times \Lambda^r \rightarrow \Lambda^r$ by

$$[\omega, \eta] = \sum_{I \in \mathcal{I}_r} \omega_I \eta_I dx_I.$$

Then

$$[\hat{\omega}(k), \lambda_k^\alpha(x)] = \sum_{I \in \mathcal{I}_r} \hat{\omega}(I, k) \ell_k^{\alpha, I}(x) dx_I.$$

The following proposition gives the inversion formula and Parseval's identity for the Fourier-Laguerre transform.

Proposition 3.8. *For every $\omega \in L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$*

$$\omega(x) = \sum_{k \in \mathbb{N}_r^d} [\hat{\omega}(k), \lambda_k^\alpha(x)],$$

where the series converges in $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$. Moreover

$$\|\omega\|_{L^2(\mu_\alpha)}^2 = \sum_{k \in \mathbb{N}_r^d} |\hat{\omega}(k)|^2.$$

Proof. If $\omega \in L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$, by definition of orthonormal basis

$$\omega(x) = \sum_{I \in \mathcal{I}_r} \sum_{k \in \mathcal{K}(I)} \hat{\omega}(I, k) \ell_k^{\alpha, I}(x) dx_I,$$

where the series converges in $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$. Since we have defined $\hat{\omega}(I, k) = 0$ for $k \notin \mathcal{K}(I)$, we may extend the sum over $\mathcal{K}(I)$ to a sum over \mathbb{N}_r^d . Thus, exchanging the sums over I and over k , we get

$$\omega(x) = \sum_{k \in \mathbb{N}_r^d} \sum_{I \in \mathcal{I}_r} \hat{\omega}(I, k) \ell_{I, k}^\alpha(x) dx_I = \sum_{k \in \mathbb{N}_r^d} [\hat{\omega}(k), \lambda_k^\alpha(x)].$$

Similarly,

$$\begin{aligned} \|\omega\|_{L^2(\mu_\alpha)}^2 &= \sum_{I \in \mathcal{I}_r} \sum_{k \in \mathcal{K}(I)} |\hat{\omega}(I, k)|^2 \\ &= \sum_{k \in \mathbb{N}_r^d} \sum_{I \in \mathcal{I}_r} |\hat{\omega}(I, k)|^2 \\ &= \sum_{k \in \mathbb{N}_r^d} |\hat{\omega}(k)|^2. \end{aligned}$$

□

Denote by $\mathcal{P}(\mathbb{R}_+^d; \Lambda^r)$ the space of finite linear combinations of elements of the basis B_r , i.e. the space of r -forms with polynomial coefficients. Clearly $\mathcal{P}(\mathbb{R}_+^d; \Lambda^r) \subset C^\infty(\mathbb{R}_+^d, \Lambda^r)$. Next, we compute the Fourier-Laguerre transforms of the forms $\delta\omega$, $\delta^*\omega$ and $\mathbb{L}_\alpha\omega$, when ω is in $\mathcal{P}(\mathbb{R}_+^d; \Lambda^r)$.

Proposition 3.9. *For every $k \in \mathbb{N}^d$ define the covector*

$$\hat{\delta}(k) = \sum_{j=1}^d \sqrt{k_j} dx_j.$$

If $\omega \in \mathcal{P}(\mathbb{R}_+^d; \Lambda^r)$ then for all $k \in \mathbb{N}_r^d$

$$(3.8) \quad \widehat{\delta\omega}(k) = \hat{\delta}(k) \wedge \hat{\omega}(k), \quad \widehat{\delta^*\omega}(k) = \iota_{\hat{\delta}(k)} \hat{\omega}(k), \quad \widehat{\mathbb{L}_\alpha\omega}(k) = |k| \hat{\omega}(k).$$

The operators δ , δ^* and \mathbb{L}_α with domain $\mathcal{P}(\mathbb{R}_+^d; \Lambda^r)$ are closable in $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$.

Proof. We observe that

$$\begin{aligned}\delta_j \ell_k^{\alpha, I} &= \sqrt{k_j} \ell_k^{\alpha, I \cup j} & \text{if } j \notin I \\ \delta_j^* \ell_k^{\alpha, I} &= \sqrt{k_j} \ell_k^{\alpha, I \setminus j} & \text{if } j \in I \\ \mathcal{L}_{\alpha, I} \ell_k^{\alpha, I} &= |k| \ell_k^{\alpha, I}.\end{aligned}$$

Indeed, the first identity follows immediately from the definition of $\ell_k^{\alpha, I}$, the second from the identity $\delta_j^* \delta_j \ell_{k_j}^{\alpha, j}(x_j) = k_j \ell_{k_j}^{\alpha, j}(x_j)$. The last identity follows from the first two and the fact that $\mathcal{L}_k^{\alpha, I} = \sum_{I \in \mathcal{I}_r} \sum_{j \in I} \delta_j \delta_j^* + \sum_{I \in \mathcal{I}_r} \sum_{j \notin I} \delta_j^* \delta_j$.

If $\omega = \sum_{I \in \mathcal{I}_r} \omega_I dx_I$ in $\mathcal{P}(\mathbb{R}_+^d; \Lambda^r)$, then

$$\delta \omega = \sum_{I \in \mathcal{I}_r} \sum_{j \notin I} \delta_j \omega_I dx_j \wedge dx_I = \sum_{I \in \mathcal{I}_r} \sum_{j \notin I} (-1)^{\sigma(j, I)} \delta_j \omega_I dx_{I \cup j}.$$

Hence

$$\begin{aligned}\widehat{\delta \omega}(I \cup j, k) &= \langle (\delta \omega)_{I \cup j}, \ell_k^{\alpha, I \cup j} \rangle_\alpha = (-1)^{\sigma(j, I)} \langle \delta_j \omega_I, \ell_k^{\alpha, I \cup j} \rangle_\alpha \\ &= (-1)^{\sigma(j, I)} \langle \omega_I, \delta_j^* \ell_k^{\alpha, I \cup j} \rangle_\alpha = (-1)^{\sigma(j, I)} \langle \omega_I, \sqrt{k_j} \ell_k^{\alpha, I} \rangle_\alpha \\ &= (-1)^{\sigma(j, I)} \sqrt{k_j} \hat{\omega}(I, k).\end{aligned}$$

Thus

$$\begin{aligned}\widehat{\delta \omega}(k) &= \sum_{J \in \mathcal{I}_{r+1}} \widehat{\delta \omega}(J, k) dx_J \\ &= \sum_{I \in \mathcal{I}_r} \sum_{j \notin I} \widehat{\delta \omega}(I \cup j, k) dx_{I \cup j} = \\ &= \sum_{I \in \mathcal{I}_r} \sum_{j \notin I} (-1)^{\sigma(j, I)} \sqrt{k_j} \hat{\omega}(I, k) dx_{I \cup j} \\ &= \sum_{I \in \mathcal{I}_r} \sum_{j \notin I} \sqrt{k_j} \hat{\omega}(I, k) dx_j \wedge dx_I \\ &= \left(\sum_{j=1}^d \sqrt{k_j} dx_j \right) \wedge \left(\sum_{I \in \mathcal{I}_r} \hat{\omega}(I, k) dx_I \right) \\ &= \hat{\delta}(k) \wedge \hat{\omega}(k).\end{aligned}$$

To prove the identity $\widehat{\delta^* \omega}(k) = \iota_{\hat{\delta}(k)} \hat{\omega}(k)$, we observe that for all $\eta \in \mathcal{P}(\mathbb{R}_+^d; \Lambda^{r-1})$

$$\begin{aligned}\langle \delta^* \omega, \eta \rangle_\alpha &= \langle \omega, \delta \eta \rangle_\alpha = \sum_{k \in \mathbb{N}^d} \langle \hat{\omega}(k), \widehat{\delta \eta}(k) \rangle_{\Lambda^r} \\ &= \sum_{k \in \mathbb{N}^d} \langle \hat{\omega}(k), \hat{\delta}(k) \wedge \hat{\eta}(k) \rangle_{\Lambda^r} \\ &= \sum_k \langle \iota_{\hat{\delta}(k)} \hat{\omega}(k), \hat{\eta}(k) \rangle_{\Lambda^{r-1}},\end{aligned}$$

by Parseval's identity and the fact that the operator of interior multiplication by $\hat{\delta}(k)$ is the adjoint with respect to the inner product on covectors of the exterior multiplication by $\hat{\delta}(k)$. Since η is arbitrary, the conclusion follows.

To prove the last identity, we observe that by Proposition 3.6 $(\mathbb{L}_\alpha \omega)_I = \mathcal{L}_{\alpha, I} \omega_I$. Thus

$$\begin{aligned}
 \widehat{\mathbb{L}_\alpha \omega}(I, k) &= \langle (\mathbb{L}_\alpha \omega)_I, \ell_k^{\alpha, I} \rangle_\alpha \\
 &= \langle \mathcal{L}_{\alpha, I} \omega_I, \ell_k^{\alpha, I} \rangle_\alpha \\
 (3.9) \quad &= \langle \omega_I, \mathcal{L}_{\alpha, I} \ell_k^{\alpha, I} \rangle_\alpha \\
 &= |k| \langle \omega_I, \ell_k^{\alpha, I} \rangle_\alpha \\
 &= |k| \widehat{\omega}(I, k).
 \end{aligned}$$

Hence

$$\widehat{\mathbb{L}_\alpha \omega}(k) = \sum_I \widehat{\mathbb{L}_\alpha \omega}(I, k) dx_I = |k| \sum_I \widehat{\omega}(I, k) dx_I = |k| \widehat{\omega}(k).$$

It is now an easy matter to see that δ , δ^* and \mathbb{L}_α with domain $\mathcal{P}(\mathbb{R}_+^d; \Lambda^r)$ are closable in $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$. Indeed, if (ω_n) is a sequence in $\mathcal{P}(\mathbb{R}_+^d; \Lambda^r)$ such that $\omega_n \rightarrow 0$ and $\delta \omega_n \rightarrow \eta$ in $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$, then $\widehat{\omega}_n(k) \rightarrow 0$ and $\widehat{\eta}(k) = \lim_n \widehat{\delta \omega}_n(k) = \widehat{\delta}(k) \wedge \widehat{\omega}_n(k) = 0$ for every k . Hence $\eta = 0$.

The proofs that δ^* and \mathbb{L}_α are closable are similar. \square

Notation 3.10. With a slight abuse of notation, we denote also by δ , δ^* and \mathbb{L}_α the closures in $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ of the operators δ , δ^* and \mathbb{L}_α on $\mathcal{P}(\mathbb{R}_+^d; \Lambda^r)$.

The following proposition characterises their domains in $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ via the Fourier-Laguerre transform.

Proposition 3.11. *The domains of δ , δ^* and \mathbb{L}_α on $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ are*

$$\begin{aligned}
 \mathcal{D}_r(\delta) &= \left\{ \omega \in L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r) : \sum_{k \in \mathbb{N}_r^d} |\widehat{\delta}(k) \wedge \widehat{\omega}(k)|_{\Lambda^{r+1}}^2 < \infty \right\} \\
 \mathcal{D}_r(\delta^*) &= \left\{ \omega \in L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r) : \sum_{k \in \mathbb{N}_r^d} |\iota_{\widehat{\delta}(k)} \widehat{\omega}(k)|_{\Lambda^{r-1}}^2 < \infty \right\} \\
 \mathcal{D}_r(\mathbb{L}_\alpha) &= \left\{ \omega \in L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r) : \sum_{k \in \mathbb{N}_r^d} |k|^2 |\widehat{\omega}(k)|_{\Lambda^r}^2 < \infty \right\}.
 \end{aligned}$$

The identities (3.8) continue to hold for ω in the domains of δ , δ^* and \mathbb{L}_α .

Proof. The proof is straightforward. \square

Proposition 3.12. *The space $C_c^\infty(\mathbb{R}_+^d; \Lambda^r)$ is contained in the spaces $\mathcal{D}_r(\delta)$, $\mathcal{D}_r(\delta^*)$ and $\mathcal{D}_r(\mathbb{L}_\alpha)$ and on it the operators δ , δ^* and \mathbb{L}_α coincide with the closures in $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ of the operators δ , δ^* and \mathbb{L}_α on $\mathcal{P}(\mathbb{R}_+^d; \Lambda^r)$.*

Proof. If $\omega \in C_c^\infty(\mathbb{R}_+^d; \Lambda^r)$ then its Fourier-Laguerre coefficients decay faster than any power of $|k|$, because by (3.9), for every positive integer m

$$\begin{aligned}
 |k|^m |\widehat{\omega}(I, k)| &= |\widehat{\mathbb{L}_\alpha^m \omega}(I, k)| \\
 &= \int_{\mathbb{R}_+^d} (\mathbb{L}_\alpha^m \omega)_I \ell_k^{\alpha, I} d\mu_\alpha \\
 &\leq \|\mathbb{L}_\alpha^m \omega\|_{L^2}.
 \end{aligned}$$

This shows that $C_c^\infty(\mathbb{R}_+^d; \Lambda^r)$ is contained in the domains $\mathcal{D}_r(\delta)$, $\mathcal{D}_r(\delta^*)$ and $\mathcal{D}_r(\mathbb{L}_\alpha)$. The fact that the operators on $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ coincide with the closures in $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ of the operators δ , δ^* and \mathbb{L}_α on $\mathcal{P}(\mathbb{R}_+^d; \Lambda^r)$ follows easily by computing the Fourier-Laguerre coefficients of $\delta\omega$, $\delta^*\omega$ and $\mathbb{L}_\alpha\omega$, intended in the classical sense, as in the proof of Proposition 3.9. \square

Proposition 3.13. *The operators δ and δ^* on their domains $\mathcal{D}_r(\delta)$ and $\mathcal{D}_r(\delta^*)$ are adjoint of each other. The operator \mathbb{L}_α on $\mathcal{D}_r(\mathbb{L}_\alpha)$ is self-adjoint and its spectral resolution is*

$$(3.10) \quad \mathbb{L}_\alpha = \sum_{n \geq r} n \mathcal{P}_n^\alpha$$

where \mathcal{P}_n^α is the orthogonal projection onto the space spanned by the forms λ_k^α , $|k| = n$.

Proof. To prove that δ^* is the adjoint of δ observe that, if $\omega \in \mathcal{D}_r(\delta)$ and $\eta \in \mathcal{D}_{r+1}(\delta^*)$, then by polarising Parseval's identity

$$\begin{aligned} \langle \delta\omega, \eta \rangle_\alpha &= \sum_{k \in \mathbb{N}_{r+1}^d} \langle \widehat{\delta\omega}(k), \hat{\eta}(k) \rangle_{\Lambda^{r+1}} \\ &= \sum_{k \in \mathbb{N}_{r+1}^d} \langle \hat{\delta}(k) \wedge \hat{\omega}(k), \hat{\eta}(k) \rangle_{\Lambda^{r+1}} \\ &= \sum_{k \in \mathbb{N}_{r+1}^d} \langle \omega(k), \iota_{\hat{\delta}(k)} \eta(k) \rangle_{\Lambda^{r+1}} \\ &= \langle \omega, \delta^* \eta \rangle_\alpha. \end{aligned}$$

Thus the adjoint of δ is an extension of δ^* .

Conversely, if $\eta \in L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^{r+1})$ is in the domain of the adjoint of δ , then for every $\omega \in \mathcal{P}(\mathbb{R}_+^d; \Lambda^r)$ there exists a constant $C(\omega)$ such that

$$|\langle \delta\omega, \eta \rangle_\alpha| \leq C(\eta) \|\omega\|_{L^2(\mu_\alpha)}.$$

This implies that

$$\left| \sum_{k \in \mathbb{N}_r^d} \langle \omega(k), \hat{\iota}_{\hat{\delta}(k)} \eta(k) \rangle_{\Lambda^r(\mathbb{R}_+^d)} \right| \leq C(\eta) \|\omega\|_{L^2(\mu_\alpha)}.$$

Since this holds for all $\omega \in \mathcal{P}(\Lambda^r(\mathbb{R}_+^d))$, it follows that

$$\sum_{k \in \mathbb{N}_r^d} \|\iota_{\hat{\delta}(k)} \eta(k)\|_{\Lambda^r(\mathbb{R}_+^d)}^2 < \infty,$$

that is $\eta \in \mathcal{D}(\delta^*)$. This proves that δ^* is the adjoint of δ . The proof that the δ is the adjoint of δ^* is similar.

To show that \mathbb{L}_α is self-adjoint, it is enough to remark that \mathbb{L}_α is unitarily equivalent, via the Fourier-Laguerre transform, to the operator of multiplication by the function $k \mapsto |k|$ acting on its natural domain in the space $\ell^2(\mathbb{N}_r^d; \Lambda^r(\mathbb{R}^d))$ of square summable $\Lambda^r(\mathbb{R}^d)$ -valued multi-sequences.

Finally the spectral resolution of \mathbb{L}_α follows from the following facts

- (a) $\{\ell_k^{\alpha, I} dx_I : I \in \mathcal{I}_r, k \in \mathcal{K}(I)\}$ is an orthonormal basis of $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$;

- (b) $\mathbb{L}_\alpha \left(\ell_k^{\alpha, I} dx_I \right) = \mathcal{L}_{I, k}^\alpha \ell_k^{\alpha, I} dx_I = |k| \ell_k^{\alpha, I} dx_I;$
- (c) $\lambda_k^\alpha = \sum_{I \in \mathcal{I}_r} \ell_k^{\alpha, I} dx_I;$
- (d) if $k \in \mathcal{K}(I)$ then $|k| \geq r.$

□

We shall denote by $\ker_r(\delta)$, $\ker_r(\delta^*)$, $\ker_r(\mathbb{L}_\alpha)$ and $\text{im}_r(\delta)$, $\text{im}_r(\delta^*)$ and $\text{im}_r(\mathbb{L}_\alpha)$ the kernels and the images of δ , δ^* and \mathbb{L}_α , considered as operators on $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$. Thus $\ker_r(\mathbb{L}_\alpha)$ is the space of r -harmonic forms in $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$. It follows from the spectral resolution of \mathbb{L}_α that the only harmonic 0-forms are the constants, while there are no non-trivial harmonic r -forms for $r \geq 1$.

In Section 3.2 we defined the Hodge-Laguerre operator on smooth forms as $\mathbb{L}_\alpha = \delta\delta^* + \delta^*\delta$. The same identity for the corresponding unbounded operators on $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ is not obvious, because one must verify that the domains of the left and right hand side coincide. Indeed, one has

Proposition 3.14. *Let*

$$\mathcal{D}_r(\delta\delta^* + \delta^*\delta) = \{\omega \in \mathcal{D}_r(\delta) \cap \mathcal{D}_r(\delta^*) : \delta\omega \in \mathcal{D}_{r+1}(\delta^*) \text{ and } \delta^*\omega \in \mathcal{D}_{r-1}(\delta)\}.$$

Then $\mathcal{D}_r(\delta\delta^ + \delta^*\delta) = \mathcal{D}_r(\mathbb{L}_\alpha)$ and $\mathbb{L}_\alpha = \delta\delta^* + \delta^*\delta$. Moreover \mathbb{L}_α commutes with δ and δ^* and*

$$\text{im}_r(\delta) \subset \ker_r(\delta), \quad \text{im}_r(\delta^*) \subset \ker_r(\delta^*),$$

i.e. $\delta^2 = 0$ and $(\delta^)^2 = 0$.*

Proof. If $\omega \in \mathcal{D}_r(\delta\delta^* + \delta^*\delta)$ then $\delta\delta^*\omega + \delta^*\delta\omega \in L^2(\Lambda^r(\mathbb{R}_+^d), \mu_\alpha)$. Hence, by Parseval's identity, the Fourier-Laguerre transform of $\delta\delta^*\omega + \delta^*\delta\omega$ is in $\ell^2(\mathbb{N}_r^d, \Lambda^r)$. Since by Lemma 3.1

$$\begin{aligned} (\delta\delta^*\omega + \delta^*\delta\omega)^\wedge(k) &= \iota_{\hat{\delta}(k)}(\hat{\delta}(k) \wedge \hat{\omega}(k)) + \hat{\delta}(k) \wedge \iota_{\hat{\delta}(k)}\hat{\omega}(k) \\ (3.11) \quad &= |\hat{\delta}(k)|^2 \hat{\omega}(k) \\ &= |k| \hat{\omega}(k), \end{aligned}$$

also the multi sequence $k \mapsto |k| \hat{\omega}(k)$ is in $\ell^2(\mathbb{N}_r^d, \Lambda^r)$, i.e. $\omega \in \mathcal{D}_r(\mathbb{L}_\alpha)$. This proves the inclusion $\mathcal{D}_r(\delta\delta^* + \delta^*\delta) \subset \mathcal{D}_r(\mathbb{L}_\alpha)$. Conversely, if $\omega \in \mathcal{D}_r(\mathbb{L}_\alpha)$, by (3.11) and Lemma 3.1

$$|\iota_{\hat{\delta}(k)}(\hat{\delta}(k) \wedge \hat{\omega}(k))|^2 + |\hat{\delta}(k) \wedge \iota_{\hat{\delta}(k)}\hat{\omega}(k)|^2 = |k|^2 |\hat{\omega}(k)|^2.$$

Thus, the same argument based on Parseval's identity proves that $\mathcal{D}_r(\mathbb{L}_\alpha) \subset \mathcal{D}_r(\delta\delta^* + \delta^*\delta)$.

The other statements can be easily proved by observing that the Fourier-Laguerre transforms of the left and the right hand sides coincide. □

Proposition 3.15.

$$\ker_r(\mathbb{L}_\alpha) = \{\omega \in \mathcal{D}_r(\delta) \cap \mathcal{D}_r(\delta^*) : \delta\omega = 0, \delta^*\omega = 0\}.$$

Proof. If $\omega \in \ker_r(\mathbb{L}_\alpha)$ then $\omega \in \mathcal{D}_r(\delta) \cap \mathcal{D}_r(\delta^*)$, by Proposition 3.14. Moreover

$$\|\delta\omega\|_{L^2(\mu_\alpha)}^2 + \|\delta^*\omega\|_{L^2(\mu_\alpha)}^2 = \langle \mathbb{L}_\alpha \omega, \omega \rangle_\alpha = 0.$$

Thus $\delta\omega = 0, \delta^*\omega = 0$. Conversely, if $\omega \in \mathcal{D}_r(\delta) \cap \mathcal{D}_r(\delta^*)$ and $\delta\omega = 0, \delta^*\omega = 0$ then, by Proposition 3.14, $\omega \in \mathcal{D}_r(\mathbb{L}_\alpha)$ and $\mathbb{L}_\alpha \omega = (\delta\delta^* + \delta^*\delta)\omega = 0$. □

In Section 3.3 we proved that the action of the Hodge-Laguerre operator on *smooth* r -forms is diagonalised by the basis $\{dx_I : I \in \mathcal{I}_r\}$ of $\Lambda^r(\mathbb{R}^d)$. A similar result holds for the action of \mathbb{L}_α on $L^2(\mu_\alpha)$ forms. Namely

Proposition 3.16. *A form $\omega = \sum_{I \in \mathcal{I}_r} \omega_I dx_I$ is in $\mathcal{D}_r(\mathbb{L}_\alpha)$ if and only if $\omega_I \in \mathcal{D}(\mathcal{L}_{\alpha,I})$ for all $I \in \mathcal{I}_r$. Moreover*

$$\mathbb{L}_\alpha \omega = \sum_{I \in \mathcal{I}_r} \mathcal{L}_{\alpha,I} \omega_I dx_I \quad \forall \omega \in \mathcal{D}_r(\mathbb{L}_\alpha).$$

Proof. The fact that ω is in $\mathcal{D}_r(\mathbb{L}_\alpha)$ if and only if $\omega_I \in \mathcal{D}(\mathcal{L}_{\alpha,I})$ for all $I \in \mathcal{I}_r$ follows immediately from the characterization of the domains via the Fourier-Laguerre coefficients of ω and of its components ω_I . The identity of the operators holds on $\mathcal{D}(\mathbb{R}_+^d; \Lambda^r)$ by Proposition 3.6 and extends to the L^2 domains of the operators, since $\mathcal{D}(\mathbb{R}_+^d; \Lambda^r)$ is dense in the domain in the graph norm. \square

We denote by $\{\mathbb{T}_t^\alpha : t \geq 0\}$ the heat semigroup on r -forms, i.e. the semigroup on $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ generated by $-\mathbb{L}_\alpha$, and by $\{\mathbb{P}_t^\alpha : t \geq 0\}$ the corresponding Poisson semigroup generated by $-\mathbb{L}_\alpha^{1/2}$. More generally, for every $\rho \leq r$ we consider the semigroups generated by $\rho I - \mathbb{L}_\alpha$ and by $-(\mathbb{L}_\alpha - \rho I)^{1/2}$, i.e. the semigroups

$$\mathbb{T}_t^{\alpha,\rho} = e^{-t(\mathbb{L}_\alpha - \rho I)}, \quad \mathbb{P}_t^{\alpha,\rho} = e^{-t(\mathbb{L}_\alpha - \rho I)^{1/2}}.$$

Their spectral resolutions are

$$(3.12) \quad \mathbb{T}_t^{\alpha,\rho} = \sum_{n \geq r} e^{-t(n-\rho)} \mathcal{P}_n^\alpha$$

$$(3.13) \quad \mathbb{P}_t^{\alpha,\rho} = \sum_{n \geq r} e^{-t\sqrt{n-\rho}} \mathcal{P}_n^\alpha,$$

where, as before, \mathcal{P}_n^α denotes the orthogonal projection onto the space spanned by the forms $\lambda_k^\alpha(x) = \sum_I \ell_k^{\alpha,I}(x) dx_I$, $|k| = n$.

Proposition 3.17. *For every $\alpha \in (-1, \infty)^d$ there exists a constant $C(\alpha)$ such that for all forms $\omega \in L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$*

- (i) $|\mathbb{T}_t^{\alpha,\rho} \omega(x)| \leq C(\alpha) e^{t(\rho-r/2)} T_t^\alpha |\omega|(x)$
- (ii) $|\mathbb{P}_t^{\alpha,\rho} \omega(x)| \leq C(\alpha) P_t^\alpha |\omega|(x)$ if $\rho \leq r/2$.

for all $x \in \mathbb{R}_+^d$ and $t \geq 0$. If $\alpha \in [-1/2, \infty)$ then $C(\alpha) = 1$.

Proof. By Proposition 3.16 $\mathbb{T}_t^\alpha \omega = \sum_{I \in \mathcal{I}_r} T_t^{\alpha,I} \omega_I$. Thus, by Proposition 2.4 and the positivity of the semigroup,

$$\begin{aligned} |\mathbb{T}_t^\alpha \omega(x)| &= \sup_{|\eta|=1} \sum_{I \in \mathcal{I}_r} \eta_I T_t^{\alpha,I} \omega_I(x) \\ &\leq \sup_{|\eta|=1} \sum_{I \in \mathcal{I}_r} |T_t^{\alpha,I}(\eta_I \omega_I)(x)| \\ &\leq C(\alpha) e^{-tr/2} \sup_{|\eta|=1} \sum_{I \in \mathcal{I}_r} T_t^\alpha |\eta_I \omega_I|(x) \\ &= C(\alpha) e^{-tr/2} T_t^\alpha \left(\sup_{|\eta|=1} \sum_{I \in \mathcal{I}_r} |\eta_I \omega_I| \right)(x) \\ &= C(\alpha) e^{-tr/2} T_t^\alpha |\omega|(x). \end{aligned}$$

This proves (i) for $\rho = 0$. The general case follows, since $\mathbb{T}_t^{\alpha, \rho} = e^{\rho t} \mathbb{T}_t^\alpha$. The estimate (ii) follows from (i) and the subordination formula

$$\mathbb{P}_t^{\alpha, \rho} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \mathbb{T}_{t^2/4u}^{\alpha, \rho} du.$$

□

Corollary 3.18. *For every $\alpha \in (-1, \infty)^d$ and $1 \leq p \leq \infty$ there exists a constant $C(\alpha, p)$ such that for all forms $\omega \in L^p \cap L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ and all $\rho \leq r$*

- (i) $\|\mathbb{T}_t^{\alpha, \rho} \omega\|_p \leq C(\alpha, p) e^{\gamma(\rho, r, p)t} \|\omega\|_p,$
- (ii) $\|\mathbb{P}_t^{\alpha, \rho} \omega\|_p \leq C(\alpha, p) \|\omega\|_p,$ if $\rho \leq r/2$,

where $\gamma(\rho, r, p) = \rho - (1 - |1/2 - 1/p|)r$.

In particular, if $\rho \leq r/2$ the semigroups $\mathbb{T}_t^{\alpha, \rho}$ and $\mathbb{P}_t^{\alpha, \rho}$ extend from $L^p \cap L^2(\Lambda^r)$ to semigroups which are uniformly bounded on $L^p(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ and for every $\delta > 0$

$$\|\mathbb{L}_\alpha^{-\delta}\|_{p-p} \leq \frac{C(\alpha, p)}{\gamma(0, r, p)} \quad \forall p \in (1, \infty).$$

Proof. By the spectral resolution (2.2), the bottom of the spectrum of \mathbb{T}_t^α on $L^2(\Lambda^r)$ is e^{-rt} . Thus we have that $\|\mathbb{T}_t^\alpha \omega\|_2 \leq e^{-rt} \|\omega\|_2$. On the other hand, by Proposition 3.17, we also have that $\|\mathbb{T}_t^\alpha \omega\|_1 \leq C(\alpha) e^{-rt/2} \|\omega\|_1$. Therefore, by interpolation, for all $1 \leq p \leq 2$,

$$(3.14) \quad \|\mathbb{T}_t^\alpha \omega\|_p \leq C(\alpha)^{|(2/p)-1|} e^{-(1-|1/2-1/p|)rt} \|\omega\|_p.$$

Since the semigroup is symmetric, by duality the same result holds also for $2 < p \leq \infty$. The result for $\mathbb{T}_t^{\alpha, \rho}$ follows immediately, since $\mathbb{T}_t^{\alpha, \rho} = e^{\rho t} \mathbb{T}_t^\alpha$. The same argument yields the desired estimate also for the Poisson semigroup $\mathbb{P}_t^{\alpha, \rho}$. The estimate of the norm of $\mathbb{L}_\alpha^{-\delta}$ follows from the identity

$$\mathbb{L}_\alpha^{-\delta} = \frac{1}{\Gamma(\delta)} \int_0^\infty t^{\delta-1} \mathbb{T}_t^\alpha dt$$

and estimate (3.14). □

4. HODGE-DE RHAM-KODAIRA DECOMPOSITION AND RIESZ-LAGUERRE TRANSFORMS

In this section we state the analogue of the classical Hodge decomposition for the Hodge-Laguerre operator and we define the Laguerre-Riesz transforms. We recall that $\ker_0(\mathbb{L}_\alpha) = \mathbb{R}$ and $\ker_r(\mathbb{L}_\alpha) = \{0\}$ if $r > 0$. Moreover we set $\text{im}_{-1}(\delta) = \{0\}$ and $\text{im}_{d+1}(\delta^*) = \{0\}$.

Theorem 4.1. *For all $r \geq 0$ the strong L^2 -Hodge decomposition holds*

$$L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r) = \ker_r(\mathbb{L}_\alpha) \oplus \text{im}_{r-1}(\delta) \oplus \text{im}_{r+1}(\delta^*).$$

Remark. Here strong refers to the fact that $\text{im}_{r-1}(\delta)$ and $\text{im}_{r+1}(\delta^*)$ are closed. The proof is essentially the same as in the classical case of complete manifolds with spectral gap (see for instance [3, Theorem 5.10]).

Definition 4.2. The Riesz transforms are the operators

$$\mathcal{R} = \delta \mathbb{L}_\alpha^{-1/2}, \quad \mathcal{R}^* = \delta^* \mathbb{L}_\alpha^{-1/2}$$

with domain $\mathcal{P}(\mathbb{R}_+^d; \Lambda^r)$.

The following proposition is the counterpart in the Laguerre setting of a classical result of Strichartz for complete Riemannian manifolds [37].

Proposition 4.3. *For $r > 0$ the Riesz transforms \mathcal{R} and \mathcal{R}^* are bounded on $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$. Moreover \mathcal{R}^* is the adjoint of \mathcal{R} .*

Proof. Let ω be a form in $\mathcal{P}(\mathbb{R}_+^d; \Lambda^r)$. Then

$$\|\delta\omega\|_2^2 + \|\delta^*\omega\|_2^2 = \langle \mathbb{L}_\alpha \omega, \omega \rangle_\alpha = \|\mathbb{L}_\alpha^{1/2} \omega\|_2^2.$$

This proves that the Riesz transforms are L^2 -bounded on $\mathcal{P}(\mathbb{R}_+^d; \Lambda^r)$. The conclusion follows, since $\mathcal{P}(\mathbb{R}_+^d; \Lambda^r)$ is dense in $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$. Since δ^* and \mathbb{L}_α commute, $\mathcal{R}^* = \mathbb{L}_\alpha^{-1/2} \delta^*$ is the adjoint of $\delta \mathbb{L}_\alpha^{-1/2}$. \square

Proposition 4.4. *For $r > 0$ the orthogonal projections onto the spaces $\text{im}_{r-1}(\delta)$ and $\text{im}_{r+1}(\delta^*)$ are*

$$\mathcal{R}\mathcal{R}^* = \delta \mathbb{L}_\alpha^{-1} \delta^*, \quad \text{and} \quad \mathcal{R}^* \mathcal{R} = \delta^* \mathbb{L}_\alpha^{-1} \delta,$$

respectively.

Proof. Let $P = \mathcal{R}\mathcal{R}^*$ and $Q = \mathcal{R}^* \mathcal{R}$. Then P and Q are bounded on $L^2(\mu_\alpha; \Lambda^r)$ and self-adjoint, by Proposition 4.3. Since δ and δ^* commute with \mathbb{L}_α , $P + Q = (\delta\delta^* + \delta^*\delta)\mathbb{L}_\alpha^{-1} = I$. Moreover $PQ = QP = 0$, because $\delta^2 = 0$ and $(\delta^*)^2 = 0$. Hence $P^2 - P = P(I - P) = PQ = 0 = (I - Q)Q = Q - Q^2$. This proves that P and Q are idempotent. Therefore they are orthogonal projections. The conclusion follows, since $\text{im}_r(P) \subseteq \text{im}_{r-1}(\delta)$ and $\text{im}_r(Q) \subseteq \text{im}_{r+1}(\delta^*)$. \square

Remark 4.5. If $r = 0$ the conclusions of Propositions 4.3 and 4.4 remain valid, if one replaces $\mathbb{L}_\alpha^{-1/2}$ and \mathbb{L}_α^{-1} by their restrictions to the orthogonal of constant functions.

Definition 4.6. More generally, for every $\rho < r$ we define the *shifted* Riesz transforms

$$\mathcal{R}_\rho = \delta(\mathbb{L}_\alpha - \rho I)^{-1/2} \quad \text{and} \quad \mathcal{R}_\rho^* = (\mathbb{L}_\alpha - \rho I)^{-1/2} \delta^*.$$

Proposition 4.7. *For every $\rho < r$ the shifted Riesz transforms \mathcal{R}_ρ and \mathcal{R}_ρ^* are bounded on $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$. Moreover \mathcal{R}_ρ^* is the adjoint of \mathcal{R}_ρ .*

Proof. As in the proof of Proposition 4.3 it suffices to observe that

$$\begin{aligned} \|\delta\omega\|_2^2 + \|\delta^*\omega\|_2^2 &= \|\mathbb{L}_\alpha^{1/2} \omega\|_2^2 \\ &= \|\mathbb{L}_\alpha^{1/2} (\mathbb{L}_\alpha - \rho I)^{-1/2} (\mathbb{L}_\alpha - \rho I)^{1/2} \omega\|_2^2 \\ &\leq \left(\frac{r}{r - \rho} \right) \|(\mathbb{L}_\alpha - \rho I)^{1/2} \omega\|_2^2 \end{aligned}$$

since $\|\mathbb{L}_\alpha^{1/2} (\mathbb{L}_\alpha - \rho I)^{-1/2}\|_2 \leq \sqrt{r/(r - \rho)}$ by (3.10). \square

5. THE BILINEAR EMBEDDING THEOREM AND ITS APPLICATIONS

We consider the manifold $M = \mathbb{R}_+^d \times \mathbb{R}_+$, with coordinates (x_1, \dots, x_d, t) . We recall that δ_i , for $i = 1, \dots, d$, denotes the Laguerre derivative $\sqrt{x_i} \partial_i$, and

we denote by $\delta_{d+1} = \partial_t$ the classical derivative with respect to t . Given a form $\omega = \sum_{I \in \mathcal{I}_r} \omega_I(x, t) dx_I$ in $C^\infty(\Lambda^r(M))$ we define

$$|\bar{\nabla} \omega(x, t)| = \left(\sum_{i=1}^{d+1} \sum_{I \in \mathcal{I}_r} |\delta_i \omega_I(x, t)|^2 \right)^{1/2}.$$

Theorem 5.1 (Bilinear embedding Theorem). *Suppose that $\alpha \in [-1/2, \infty)^d$ and $\rho \leq r/2$. For each $p \in (1, \infty)$, $\omega \in \mathcal{P}(\Lambda^r(M))$ and $\eta \in \mathcal{P}(\Lambda^{r+1}(M))$*

$$\int_0^\infty \int_{\mathbb{R}_+^d} |\bar{\nabla} \mathbb{P}_t^{\alpha, \rho} \omega(x)| |\bar{\nabla} \mathbb{P}_t^{\alpha, \rho} \eta(x)| d\mu_\alpha(x) t dt \leq 6(p^* - 1) \|\omega\|_{L^p(\mu_\alpha)} \|\eta\|_{L^q(\mu_\alpha)},$$

where q is the conjugate exponent of p , and $p^* = \max\{p, q\}$.

We postpone the proof of this result to deduce some of its consequences.

5.1. Riesz-Laguerre transforms on $L^p(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$. A first consequence of the Bilinear Embedding Theorem is the boundedness on $L^p(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ of the shifted Riesz transforms \mathcal{R}_ρ when $\rho \leq r/2$.

Theorem 5.2. *Suppose that $\alpha \in [-1/2, \infty)^d$, $r \geq 1$ and $\rho \leq r/2$. Then for each $p \in (1, \infty)$ the shifted Riesz transforms \mathcal{R}_ρ and \mathcal{R}_ρ^* extend to bounded operators from $L^p(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ to $L^p(\mathbb{R}_+^d, \mu_\alpha; \Lambda^{r+1})$ and to $L^p(\mathbb{R}_+^d, \mu_\alpha; \Lambda^{r-1})$, respectively. Moreover for all $\omega \in L^p(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$*

$$\|\mathcal{R}_\rho \omega\|_{L^p(\mu_\alpha)} \leq C(p) \|\omega\|_{L^p(\mu_\alpha)}, \quad \|\mathcal{R}_\rho^* \omega\|_{L^p(\mu_\alpha)} \leq C(p) \|\omega\|_{L^p(\mu_\alpha)}$$

where $C(p) = 24(p^* - 1)$. If $r = 0$ the inequality holds for all ω in $L^p(\mathbb{R}_+^d, \mu_\alpha; \Lambda^0)$ with integral zero.

Proof of Theorem 5.2. The result is a straightforward consequence of Theorem 5.1, the following representation formula and a duality argument.

Lemma 5.3. *If $r \geq 1$ then for every $\omega \in \mathcal{P}(\Lambda^r(\mathbb{R}_+^d))$ and $\eta \in \mathcal{P}(\Lambda^{r+1}(\mathbb{R}_+^d))$,*

$$(5.1) \quad \langle \mathcal{R}_\rho \omega, \eta \rangle_\alpha = -4 \int_0^\infty \left\langle \delta \mathbb{P}_t^{\alpha, \rho} \omega, \frac{d}{dt} \mathbb{P}_t^{\alpha, \rho} \eta \right\rangle_\alpha t dt.$$

If $r = 0$ the identity holds for all ω in $\mathcal{P}(\Lambda^0(\mathbb{R}_+^d))$ orthogonal to the constants.

Proof of Lemma 5.3. Let

$$\Psi(t) = \langle \mathbb{P}_t^{\alpha, \rho} \mathcal{R} \omega, \mathbb{P}_t^{\alpha, \rho} \eta \rangle_\alpha.$$

We claim that

- (i) $\lim_{t \rightarrow \infty} \Psi(t) = \lim_{t \rightarrow \infty} t \Psi'(t) = 0$.
- (ii) $\Psi'(t) = -2 \langle \delta \mathbb{P}_t^{\alpha, \rho} \omega, \mathbb{P}_t^{\alpha, \rho} \eta \rangle_\alpha$;
- (iii) $\Psi''(t) = -4 \langle \delta \mathbb{P}_t^{\alpha, \rho} \omega, \frac{d}{dt} \mathbb{P}_t^{\alpha, \rho} \eta \rangle_\alpha$;

Assuming the claim for the moment, the desired identity follows since

$$\langle \mathcal{R} \omega, \eta \rangle_\alpha = \Psi(0) = \int_0^\infty \Psi''(t) t dt = -4 \int_0^\infty \langle \delta \mathbb{P}_t^{\alpha, \rho} \omega, \frac{d}{dt} \mathbb{P}_t^{\alpha, \rho} \eta \rangle_\alpha t dt.$$

It remains only to prove the claim. By the spectral resolution of the Poisson semigroup (3.13)

$$\Psi(t) = \sum_{|k| \geq r} e^{-2t|k-\rho|^{1/2}} \langle \mathcal{P}_k \mathcal{R} \omega, \eta \rangle_\alpha.$$

This proves (i) (notice that if $r = 0$ the sum starts from 1, since we assume that ω is orthogonal to the constants). The other two identities follow easily, since δ and $\mathcal{R}_\rho = \delta(\mathbb{L}_\alpha - \rho I)^{-1/2}$ commute with $\mathbb{P}_t^{\alpha, \rho}$ and

$$\frac{d}{dt} \mathbb{P}_t^{\alpha, \rho} \omega = -(\mathbb{L}_\alpha - \rho I)^{1/2} \mathbb{P}_t^{\alpha, \rho} \omega, \quad \mathcal{R}_\rho(\mathbb{L}_\alpha - \rho I)^{1/2} \omega = \delta \omega \quad \forall \omega \in \mathcal{P}(\Lambda^r(\mathbb{R}_+^d)).$$

Indeed, since $\Psi(t) = \langle \mathcal{R}_\rho \mathbb{P}_{2t}^{\alpha, \rho} \omega, \eta \rangle$,

$$\begin{aligned} \Psi'(t) &= -2 \langle \delta \mathbb{P}_{2t}^{\alpha, \rho} \omega, \eta \rangle_\alpha = -2 \langle \delta \mathbb{P}_t^{\alpha, \rho} \omega, \mathbb{P}_t^{\alpha, \rho} \eta, \rangle_\alpha \\ \Psi''(t) &= 4 \langle \delta(\mathbb{L}_\alpha - \rho I)^{1/2} \mathbb{P}_{2t}^{\alpha, \rho} \omega, \eta \rangle_\alpha = 4 \langle \delta \mathbb{P}_t^{\alpha, \rho} \omega, (\mathbb{L}_\alpha - \rho I)^{1/2} \mathbb{P}_t^{\alpha, \rho} \eta, \rangle_\alpha \\ &= -4 \left\langle \delta \mathbb{P}_t^{\alpha, \rho} \omega, \frac{d}{dt} \mathbb{P}_t^{\alpha, \rho} \eta, \right\rangle_\alpha. \end{aligned}$$

□

□

5.2. The Hodge decomposition for $L^p(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$. In this subsection we prove the strong Hodge decomposition for $L^p(\mu_\alpha; \Lambda^r)$ for all $p \in (1, \infty)$. First we define the Sobolev space $W^{1,p}(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$. To simplify the notation we shall write $L^p(\mu_\alpha; \Lambda^r)$ instead of $L^p(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ and $C_c^\infty(\Lambda^r)$ instead of $C_c^\infty(\mathbb{R}_+^d; \Lambda^r)$.

For every $p \in (1, \infty)$ denote by H_p the operator from $L^p(\mu_\alpha; \Lambda^r)$ to $L^p(\mu_\alpha; \Lambda^{r+1}) \times L^p(\mu_\alpha; \Lambda^{r-1})$ defined by

$$\omega \mapsto H_p \omega = (\delta \omega, \delta^* \omega)$$

with domain the space $\mathcal{P}(\mathbb{R}_+^d; \Lambda^r)$ of polynomial forms.

Lemma 5.4. *The operator H_p is closable.*

Proof. Suppose that (ω_n) is a sequence in $\mathcal{P}(\mathbb{R}_+^d; \Lambda^r)$ such that $\omega_n \rightarrow 0$ in $L^p(\mu_\alpha; \Lambda^r)$ and $H_p \omega_n \rightarrow (\phi, \psi)$ in $L^p(\mu_\alpha; \Lambda^{r+1}) \times L^p(\mu_\alpha; \Lambda^{r-1})$. Then $\delta \omega_n \rightarrow \phi$ in $L^p(\mu_\alpha; \Lambda^{r+1})$ and $\delta^* \omega_n \rightarrow \psi$ in $L^p(\mu_\alpha; \Lambda^{r-1})$. Therefore, for $\eta \in C_c^\infty(\Lambda^{r+1})$ and every $\zeta \in C_c^\infty(\Lambda^{r-1})$

$$\langle \phi, \eta \rangle_\alpha = \lim_{n \rightarrow \infty} \langle \delta \omega_n, \eta \rangle_\alpha = \lim_{n \rightarrow \infty} \langle \omega_n, \delta^* \eta \rangle_\alpha = 0,$$

and

$$\langle \psi, \zeta \rangle_\alpha = \lim_{n \rightarrow \infty} \langle \delta^* \omega_n, \zeta \rangle_\alpha = \lim_{n \rightarrow \infty} \langle \omega_n, \delta \zeta \rangle_\alpha = 0,$$

Hence $(\phi, \psi) = (0, 0)$ and H_p is closable. □

Definition 5.5. We define the Sobolev space $W^{1,p}(\Lambda^r) = W^{1,p}(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ as the domain of the closure \overline{H}_p of the operator H_p endowed with the graph norm.

If $\omega \in W^{1,p}(\Lambda^r)$, and $\overline{H}_p \omega = (\phi, \psi)$, with a slight abuse of notation, we shall write $\delta \omega = \phi$ and $\delta^* \omega = \psi$. Thus, $W^{1,p}(\Lambda^r)$ is the space of forms $\omega \in L^p(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ such that $\delta \omega \in L^p(\mu_\alpha; \Lambda^{r+1})$, $\delta^* \omega \in L^p(\mu_\alpha; \Lambda^{r-1})$ and

$$\|\omega\|_{W^{1,p}} = \|\omega\|_{L^p(\mu_\alpha)} + \|\delta \omega\|_{L^p(\mu_\alpha)} + \|\delta^* \omega\|_{L^p(\mu_\alpha)}.$$

Proposition 5.6. *For $1 < p < \infty$ the space $W^{1,p}(\Lambda^r)$ is reflexive.*

Proof. The space $W^{1,p}(\Lambda^r)$ is isometrically isomorphic to the graph $G(\overline{H}_p)$ of the operator \overline{H}_p . Since $G(\overline{H}_p)$ is closed in $L^p(\mu_\alpha; \Lambda^r) \times L^p(\mu_\alpha; \Lambda^{r+1}) \times L^p(\mu_\alpha; \Lambda^{r-1})$ and the latter space is reflexive, $G(\overline{H}_p)$, and hence $W^{1,p}(\Lambda^r)$, is reflexive. □

For every $p \in (1, \infty)$ denote by $\mathbb{L}_{\alpha,p}$ the infinitesimal generator of the semigroup $\{\mathbb{T}_t^\alpha : t \geq 0\}$ on $L^p(\mu_\alpha; \Lambda^r)$. Then the operators $\mathbb{L}_{\alpha,p}^{-1/2}$ and $\mathbb{L}_{\alpha,p}^{-1}$ are bounded on $L^p(\mu_\alpha; \Lambda^r)$, by Corollary 3.18. Since for all $1 < p, q < \infty$ the operators $\mathbb{L}_{\alpha,p}$ are consistent, i.e. $\mathbb{L}_{\alpha,p} = \mathbb{L}_{\alpha,q}$ on $L^p \cap L^q(\mu_\alpha; \Lambda^r)$, to simplify notation henceforth we shall simply write \mathbb{L}_α instead of $\mathbb{L}_{\alpha,p}$.

Lemma 5.7. *Suppose that $\alpha \in [-1/2, \infty)^d$. If $\omega \in L^p(\mu_\alpha; \Lambda^r)$ then $\mathbb{L}_\alpha^{-1}\omega$, $\delta\mathbb{L}_\alpha^{-1}\omega$ and $\delta^*\mathbb{L}_\alpha^{-1}\omega$ are in $W^{1,p}(\Lambda^r)$. Moreover*

$$(5.2) \quad (\delta\delta^* + \delta^*\delta)\mathbb{L}_\alpha^{-1}\omega = \omega.$$

Proof. Since by Theorem 5.2 the Riesz transforms $\delta\mathbb{L}_\alpha^{-1/2}$ and $\delta^*\mathbb{L}_\alpha^{-1/2}$ are bounded on $L^p(\Lambda^r)$, the operators $\delta\mathbb{L}_\alpha^{-1} = \delta\mathbb{L}_\alpha^{-1/2}\mathbb{L}_\alpha^{-1/2}$ and $\delta^*\mathbb{L}_\alpha^{-1} = \delta^*\mathbb{L}_\alpha^{-1/2}\mathbb{L}_\alpha^{-1/2}$ are bounded from $L^p(\Lambda^r)$ to $L^p(\Lambda^{r+1})$ and from $L^p(\Lambda^r)$ to $L^p(\Lambda^{r-1})$, and the operators $\delta\delta^*\mathbb{L}_\alpha^{-1} = \delta\mathbb{L}_\alpha^{-1/2}\delta^*\mathbb{L}_\alpha^{-1/2}$, $\delta^*\delta\mathbb{L}_\alpha^{-1} = \delta^*\mathbb{L}_\alpha^{-1/2}\delta\mathbb{L}_\alpha^{-1/2}$ are bounded on $L^p(\Lambda^r)$. Thus, if (ω_n) is a sequence in $\mathcal{P}(\mathbb{R}_+^d; \Lambda^r)$ that converges to ω in $L^p(\mu_\alpha; \Lambda^r)$, then the sequences $(\mathbb{L}_\alpha^{-1}\omega_n)$, $(\delta\mathbb{L}_\alpha^{-1}\omega_n)$, $(\delta^*\mathbb{L}_\alpha^{-1}\omega_n)$, $(\delta\delta^*\mathbb{L}_\alpha^{-1}\omega_n)$ and $(\delta^*\delta\mathbb{L}_\alpha^{-1}\omega_n)$ converge in $L^p(\mu_\alpha; \Lambda^r)$. The conclusion follows, since $W^{1,p}(\Lambda^r)$ is by definition the domain of the closure of the operator $\omega \mapsto (\delta\omega, \delta^*\omega)$ in $L^p(\mu_\alpha; \Lambda^r)$. Finally (5.2) follows from a density argument, since the identity holds on $\mathcal{P}(\mathbb{R}_+^d; \Lambda^r)$ and the operator $(\delta\delta^* + \delta^*\delta)\mathbb{L}_\alpha^{-1}$ is bounded on $L^p(\mu_\alpha; \Lambda^r)$. \square

The following result is the strong Hodge decomposition in $L^p(\mu_\alpha; \Lambda^r)$.

Theorem 5.8. *Suppose that $\alpha \in [-1/2, \infty)^d$. For every $p \in (1, \infty)$*

$$L^p(\mu_\alpha; \Lambda^r) = \delta W^{1,p}(\Lambda^{r-1}) \oplus \delta^* W^{1,p}(\Lambda^{r+1}) \quad \forall r = 1, \dots, d.$$

Moreover the spaces $\delta W^{1,p}(\Lambda^{r-1})$ and $\delta^ W^{1,p}(\Lambda^{r+1})$ are closed in $L^p(\mu_\alpha; \Lambda^r)$.*

Proof. For every $\omega \in \mathcal{P}(\mathbb{R}_+^d; \Lambda^r)$

$$\omega = \mathbb{L}_\alpha \mathbb{L}_\alpha^{-1} \omega = (\delta\delta^* + \delta^*\delta)\mathbb{L}_\alpha^{-1} \omega = \delta\delta^*\mathbb{L}_\alpha^{-1} \omega + \delta^*\delta\mathbb{L}_\alpha^{-1} \omega.$$

Since $\mathcal{P}(\mathbb{R}_+^d; \Lambda^r)$ is dense in $L^p(\mu_\alpha; \Lambda^r)$ and the operators $\delta\delta^*\mathbb{L}_\alpha^{-1}$ and $\delta^*\delta\mathbb{L}_\alpha^{-1}$ are bounded on $L^p(\mu_\alpha; \Lambda^r)$, the same identity holds for $\omega \in L^p(\mu_\alpha; \Lambda^r)$. Since $\mathbb{L}_\alpha^{-1}L^p(\mu_\alpha; \Lambda^r) \subset W^{1,p}(\Lambda^r)$ by Lemma 5.7, it holds that

$$L^p(\mu_\alpha; \Lambda^r) = \delta W^{1,p}(\Lambda^{r-1}) + \delta^* W^{1,p}(\Lambda^{r+1}).$$

To prove that the sum is direct, observe that if we write $P = \delta\delta^*\mathbb{L}_\alpha^{-1}$ and $Q = \delta^*\delta\mathbb{L}_\alpha^{-1}$, then $P + Q = I$ and $PQ = QP = 0$, since these identities hold on $\mathcal{P}(\mathbb{R}_+^d; \Lambda^r)$ and P and Q are bounded on $L^p(\mu_\alpha; \Lambda^r)$. Moreover, if $\omega \in \delta W^{1,p}(\mathbb{R}_+^d; \mu_\alpha; \Lambda^{r-1}) \cap W^{1,p}(\Lambda^{r+1})$ then $P\omega = \delta\delta^*\mathbb{L}_\alpha^{-1}\delta^*\psi = 0$ and $Q\omega = \delta^*\delta\mathbb{L}_\alpha^{-1}\delta\phi = 0$. Thus $\omega = 0$. Here we have used the fact that $\delta^*\mathbb{L}_\alpha^{-1}\delta^* = 0$ and $\delta\mathbb{L}_\alpha^{-1}\delta = 0$, because these operators are bounded on $L^p(\mu_\alpha; \Lambda^r)$ and vanish on $\mathcal{P}(\mathbb{R}_+^d; \Lambda^r)$, which is dense in $L^p(\mu_\alpha; \Lambda^r)$.

It remains only to show that $\delta W^{1,p}(\Lambda^{r-1})$ and $\delta^* W^{1,p}(\Lambda^{r+1})$ are closed subspaces of $L^p(\mu_\alpha; \Lambda^r)$. If $\omega \in \overline{\delta W^{1,p}(\Lambda^{r-1})}$, then, since $\mathcal{P}(\mathbb{R}_+^d; \Lambda^r)$ is dense in $W^{1,p}(\Lambda^{r-1})$ and δ is continuous from $W^{1,p}(\Lambda^{r-1})$ to $L^p(\mu_\alpha; \Lambda^r)$, there exists a sequence (η_j) in $\mathcal{P}(\mathbb{R}_+^d; \Lambda^{r-1})$ such that $\delta\eta_j \rightarrow \omega$ in $L^p(\mu_\alpha; \Lambda^r)$. Since $\eta_j \in \mathcal{P}(\mathbb{R}_+^d; \Lambda^{r-1})$, we can write

$$\eta_j = \delta\delta^*\mathbb{L}_\alpha^{-1}\eta_j + \delta^*\delta\mathbb{L}_\alpha^{-1}\eta_j = \beta_j + \gamma_j.$$

Observe that $\delta\beta_j = 0$ since $\text{im}(\delta) \subseteq \ker(\delta)$; thus $\delta\gamma_j = \delta\eta_j \in L^p(\mu_\alpha; \Lambda^r)$. Moreover, also γ_j is in $L^p(\mu_\alpha; \Lambda^{r-1})$ since $\gamma_j = \delta^*\delta\mathbb{L}_\alpha^{-1}\eta_j$ and the operator $\delta^*\delta\mathbb{L}_\alpha^{-1}$ is bounded on $L^p(\mu_\alpha; \Lambda^r)$. Thus γ_j is in $W^{1,p}(\Lambda^{r-1})$. Therefore (γ_j) is a bounded sequence in $W^{1,p}(\Lambda^{r-1})$. Since $W^{1,p}(\Lambda^{r-1})$ is reflexive, there exists a subsequence (γ_{j_k}) that converges to some $\gamma \in W^{1,p}(\Lambda^{r-1})$ in the weak topology. Since the norm $\|\cdot\|_{L^p(\mu_\alpha)}$ is L^p -weakly-lower-semi-continuous, and the operator δ is continuous from $W^{1,p}(\Lambda^{r-1})$ to $L^p(\mu_\alpha; \Lambda^r)$ in the weak topologies of both spaces,

$$\begin{aligned} \|\delta\gamma - \omega\|_{L^p(\mu_\alpha)} &\leq \liminf_{j \rightarrow \infty} \|\delta\gamma_j - \omega\|_{L^p(\mu_\alpha)} \\ &= \liminf_{j \rightarrow \infty} \|\delta\eta_j - \omega\|_{L^p(\mu_\alpha)} = 0. \end{aligned}$$

This shows that ω is in $\delta W^{1,p}(\Lambda^{r-1})$. Thus $\delta W^{1,p}(\Lambda^{r-1})$ is closed in $L^p(\mu_\alpha; \Lambda^r)$. The proof that also $\delta^* W^{1,p}(\Lambda^{r+1})$ is closed is similar. \square

5.3. The Hodge system and the de Rham equation in L^p . Throughout this subsection we assume that $\alpha \in [-1/2, \infty)^d$. We discuss here an existence theorem of the Hodge system associated to the Laguerre operator. The operator $\delta : \mathcal{P}(\Lambda^r) \rightarrow L^p(\mathbb{R}_+^d, \mu_\alpha; \Lambda^{r+1})$ is closable in the L^p -norm. Denote by $\Omega^p(\Lambda^r)$ the domain of its closure. As usual, we shall abuse notation denoting by δ also the closure. Notice that $\delta\mathbb{L}_\alpha^{-1/2}\varphi = \mathbb{L}_\alpha^{-1/2}\delta\varphi$ for every $\varphi \in \Omega^p(\Lambda^r)$, by a density argument, since the operators $\delta\mathbb{L}_\alpha^{-1/2}$ and $\mathbb{L}_\alpha^{-1/2}\delta$ coincide on $\mathcal{P}(\Lambda^r)$ and are bounded from $\Omega^p(\Lambda^r)$ to $L^p(\mathbb{R}_+^d, \mu_\alpha; \Lambda^{r+1})$.

Similarly, if we denote by $\Omega_*^p(\Lambda^r)$ the domain of the closure in the L^p -norm of the operator $\delta^* : \mathcal{P}(\Lambda^r) \rightarrow L^p(\mathbb{R}_+^d, \mu_\alpha; \Lambda^{r-1})$, then $\delta^*\mathbb{L}_\alpha^{-1/2}\varphi = \mathbb{L}_\alpha^{-1/2}\delta^*\varphi$ for all $\varphi \in \Omega_*^p(\Lambda^r)$.

Theorem 5.9. *For every $p \in (1, \infty)$ and $r = 1, \dots, d$, and for all $\varphi \in \Omega^p(\Lambda^{r+1})$, $\psi \in \Omega_*^p(\Lambda^{r-1})$ such that*

$$\delta\varphi = 0 \quad \text{and} \quad \delta^*\psi = 0,$$

there exists a unique $\omega \in W^{1,p}(\Lambda^r)$ solving the Hodge system

$$\delta\omega = \varphi \quad \text{and} \quad \delta^*\omega = \psi.$$

Moreover, there exists a constant $C_{p,r} > 0$ such that

$$\|\omega\|_{L^p(\mu_\alpha)} \leq C_{\alpha,p,r} (\|\varphi\|_{L^p(\mu_\alpha)} + \|\psi\|_{L^p(\mu_\alpha)}).$$

Proof. Since the operator \mathbb{L}_α is invertible on $L^p(\mu_\alpha; \Lambda^r)$, we may define

$$\omega = \delta^*\mathbb{L}_\alpha^{-1}\varphi + \delta\mathbb{L}_\alpha^{-1}\psi.$$

Then, $\omega \in W^{1,p}(\Lambda^r)$ by Lemma 5.7 and, since $\delta\mathbb{L}_\alpha^{-1}\varphi = \mathbb{L}_\alpha^{-1}\delta\varphi = 0$,

$$\begin{aligned} \delta\omega &= \delta\delta^*\mathbb{L}_\alpha^{-1}\varphi \\ &= (\delta\delta^* + \delta^*\delta)\mathbb{L}_\alpha^{-1}\varphi \\ &= \mathbb{L}_\alpha^{-1}\mathbb{L}_\alpha\varphi \\ &= \varphi, \end{aligned}$$

In a similar way one can show that $\delta^* \omega = \psi$. Moreover, by the L^p -boundedness of the Riesz transforms and of $\mathbb{L}_\alpha^{-1/2}$ (see Corollary 3.18),

$$\begin{aligned} \|\omega\|_{L^p(\mu_\alpha)} &\leq \|\delta^* \mathbb{L}_\alpha^{-\frac{1}{2}}\|_{p-p} \|\mathbb{L}_\alpha^{-\frac{1}{2}} \varphi\|_{L^p(\mu_\alpha)} + \|\delta \mathbb{L}_\alpha^{-\frac{1}{2}}\|_{p-p} \|\mathbb{L}_\alpha^{-\frac{1}{2}} \psi\|_{L^p(\mu_\alpha)} \\ &= C_{\alpha,p,r} (\|\varphi\|_{L^p(\mu_\alpha)} + \|\psi\|_{L^p(\mu_\alpha)}). \end{aligned}$$

This shows also that the solution is unique. \square

As a last application of the Bilinear Embedding Theorem, we give an existence theorem of the de Rham equation.

Theorem 5.10. *For every $p \in (1, \infty)$ and $r = 1, \dots, d$, and for all $\varphi \in \Omega^p(\Lambda^r)$ such that $\delta\varphi = 0$, there exists $\omega \in W^{1,p}(\Lambda^{r-1})$ solving the de Rham equation*

$$\delta\omega = \varphi,$$

and satisfying the estimate

$$\|\omega\|_{L^p(\mu_\alpha)} \leq C_{\alpha,p,r} \|\varphi\|_{L^p(\mu_\alpha)}.$$

Proof. It suffices to apply Theorem 5.9 with $\psi = 0$. \square

6. BELLMAN FUNCTION

The Bellman function technique was introduced in harmonic analysis by Nazarov, Treil and Volberg in [28]. We recall here the definition and the basic properties of the particular Bellman function used by A. Carbonaro and O. Dragičević in [4] to prove the boundedness of Riesz transforms on Riemannian manifolds. Even though the results coincide with those in [4] we have included full proofs for completeness.

Assume that $p \geq 2$ and let $q = \frac{p}{p-1}$ be the conjugate exponent of p ; moreover set $\gamma = \frac{q(q-1)}{8}$. We define the function $\beta : \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \rightarrow \overline{\mathbb{R}}_+$ by

$$(6.1) \quad \beta(u, v) = u^p + v^q + \gamma \begin{cases} u^2 v^{2-q} & \text{if } u^p \leq v^q \\ \frac{2}{p} u^p + \left(\frac{2}{q} - 1\right) v^q & \text{if } u^p > v^q. \end{cases}$$

The particular *Bellman function* we are going to use is the map

$$Q : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$$

defined by

$$Q(\xi, \eta) = \frac{1}{2} \beta(|\xi|, |\eta|).$$

This function is an adaptation of the one introduced by Nazarov and Treil in [29]. The proof of the following lemma is straightforward.

Lemma 6.1. *The function β is C^1 on its domain and it is C^2 except on the set $\{(u, v) : u^p = v^q \text{ or } v = 0\}$. Moreover, for every $u, v \geq 0$*

- (i) $0 \leq \beta(u, v) \leq (1 + \gamma)(u^p + v^q)$;
- (ii) *there exists a constant C such that*

$$0 \leq \partial_u \beta(u, v) \leq C p \max\{u^{p-1}, v\} \quad \text{and} \quad 0 \leq \partial_v \beta(u, v) \leq C v^{q-1}.$$

If $\zeta = (\xi, \eta)$ and $z = (x, y) \in \mathbb{R}^m \times \mathbb{R}^n$ we denote by $H_Q(z)$ the Hessian matrix of Q at ζ and by

$$H_Q(\zeta; z) = \langle H_Q(\zeta)z, z \rangle_{\mathbb{R}^m \times \mathbb{R}^n}$$

the corresponding Hessian form.

Proposition 6.2. *The function Q is in $C^1(\mathbb{R}^m \times \mathbb{R}^n)$, and it is in C^2 everywhere in $\mathbb{R}^m \times \mathbb{R}^n$ except on the set*

$$\Upsilon = \{(\xi, \eta) \in \mathbb{R} \times \mathbb{R}^d : \eta = 0 \text{ or } |\xi|^p = |\eta|^q\}.$$

If $\zeta = (\xi, \eta) \in \mathbb{R}^m \times \mathbb{R}^n \setminus \Upsilon$ then there exists $\tau = \tau(|\xi|, |\eta|)$ such that

$$H_Q(\zeta; z) \geq \frac{\gamma}{2} (\tau|x|^2 + \tau^{-1}|y|^2) \quad \forall z = (x, y) \in \mathbb{R}^m \times \mathbb{R}^n.$$

Proof. By the chain rule the regularity properties of Q follow from those of β in Lemma 6.1 and the fact that since $p \geq 2$ and $q > 1$ the function $\xi \mapsto |\xi|^p$ is in C^2 everywhere and $\eta \mapsto |\eta|^q$ is in C^1 everywhere and in C^2 in $\mathbb{R}^m \times \mathbb{R}^n \setminus \Upsilon$.

It remains to prove the estimate of the Hessian form. We observe that $H_Q(\zeta; z)$ is the sum of three forms, i.e.

$$H_Q(\zeta; z) = \sum_{i,j=1}^m \partial_{\xi_i \xi_j} Q(z) x_i x_j + 2 \sum_{i=1}^m \sum_{j=1}^n \partial_{\xi_i \eta_j} Q(z) x_i y_j + \sum_{i,j=1}^n \partial_{\eta_i \eta_j} Q(z) y_i y_j$$

that we must estimate in each of the two regions

$$R_1 = \{(\xi, \eta) : |\xi|^p < |\eta|^q, \eta \neq 0\} \quad \text{and} \quad R_2 = \{(\xi, \eta) : |\xi|^p > |\eta|^q, \eta \neq 0\}.$$

First we compute the derivatives of Q in R_1 . Since in R_1

$$Q(\xi, \eta) = \frac{1}{2} (|\xi|^p + |\eta|^q + \gamma |\xi|^2 |\eta|^{2-q}),$$

we have that

$$\begin{aligned} \partial_{\xi_i \xi_j}^2 Q(\zeta) &= \frac{1}{2} \{p(p-2)|\xi|^{p-4} \xi_i \xi_j + (p|\xi|^{p-2} + 2\gamma|\eta|^{2-q}) \delta_{ij}\} \\ \partial_{\xi_i \eta_j}^2 Q(\zeta) &= \gamma(2-q)|\eta|^{-q} \xi_i \eta_j \\ \partial_{\eta_i \eta_j}^2 Q(\xi, \eta) &= \frac{q}{2} |\eta|^{q-2} \{(q-2)|\eta|^{-2} \eta_i \eta_j + \delta_{ij}\} \\ &\quad + \frac{\gamma}{2} (2-q) |\xi|^2 |\eta|^{-q} \{-q|\eta|^{-2} \eta_i \eta_j + \delta_{ij}\}. \end{aligned}$$

Thus, in R_1

$$\begin{aligned} \sum_{i,j=1}^m \partial_{\xi_i \xi_j}^2 Q(\zeta) x_i x_j &= \frac{1}{2} p(p-2) |\xi|^{p-4} \langle \xi, x \rangle^2 + \frac{1}{2} (p|\xi|^{p-2} + 2\gamma|\eta|^{2-q}) |x|^2 \\ &\geq \gamma |\eta|^{2-q} |x|^2. \end{aligned}$$

Next, we have that

$$\begin{aligned} 2 \sum_{i=1}^m \sum_{j=1}^n \partial_{\xi_i \eta_j}^2 Q(\zeta) x_i y_j &= 2\gamma(2-q) |\eta|^{-q} \langle \xi, x \rangle \langle \eta, y \rangle \\ &\geq -2\gamma(2-q) |\eta|^{-q} |\xi| |x| |\eta| |y| \\ &\geq -2\gamma |x| |y| \\ &\geq -\gamma \left(\frac{|\eta|^{2-q} |x|^2}{2} + 2|\eta|^{q-2} |y|^2 \right); \end{aligned}$$

where, in the third inequality, we have used the fact that $|\xi| |\eta|^{1-q} \leq 1$ in R_1 . Finally, recalling that $\gamma = q(q-1)/8$,

$$\begin{aligned} \sum_{i,j=1}^d \partial_{\eta_i \eta_j}^2 Q(\zeta) y_i y_j &= \frac{q}{2} |\eta|^{q-2} \{ (q-2) |\eta|^{-2} \langle \eta, y \rangle^2 + |y|^2 \} \\ &\quad + \frac{\gamma}{2} (2-q) |\xi|^2 |\eta|^{-q} \{ -q |\eta|^{-2} \langle \eta, y \rangle^2 + |y|^2 \} \\ &\geq \frac{\gamma}{2} \{ 8 |\eta|^{q-2} + (2-q)(1-q) |\xi|^2 |\eta|^{-q} \} |y|^2 \\ &\geq \frac{\gamma}{2} [8 + (2-q)(1-q)] |\eta|^{q-2} |y|^2 \end{aligned}$$

where, in the second inequality, we have used the fact that $|\xi|^2 |\eta|^{-q} \leq |\eta|^{2-q}$ in R_1 . Combining these estimates of the three forms, we obtain that

$$\begin{aligned} H_Q(\zeta; z) &\geq \frac{\gamma}{2} (|\eta|^{2-q} |x|^2 + (q^2 - 3q + 6) |\eta|^{q-2} |y|^2) \\ &\geq \frac{\gamma}{2} (|\eta|^{2-q} |x|^2 + |\eta|^{q-2} |y|^2) \\ &\geq \frac{\gamma}{2} (\tau |x|^2 + \tau^{-1} |y|^2), \end{aligned}$$

with $\tau = |\eta|^{q-2}$.

Next, we estimate the Hessian form of Q in the region R_2 . Since in R_2

$$Q(\zeta) = \frac{1}{2} \left[|\xi|^p + |\eta|^q + \gamma \left(\frac{2}{p} |\xi|^p + \left(\frac{2}{q} - 1 \right) |\eta|^q \right) \right],$$

the second derivatives of Q are:

$$\begin{aligned} \partial_{\xi_i \xi_j}^2 Q(\zeta) &= \frac{1}{2} (p + 2\gamma) |\xi|^{p-2} \left[(p-2) \frac{\xi_i \xi_j}{|\xi|^2} + \delta_{ij} \right] \\ \partial_{\xi_i \eta_j}^2 Q(\zeta) &= 0 \\ \partial_{\eta_i \eta_j}^2 Q(\zeta) &= \frac{1}{2} (q + \gamma(2-q)) |\eta|^{q-2} \left[(q-2) \frac{\eta_i \eta_j}{|\eta|^2} + \delta_{ij} \right]. \end{aligned}$$

Hence

$$\begin{aligned} H_Q(\zeta; z) &= \sum_{i,j=1}^m \partial_{\xi_i \xi_j}^2 Q(\zeta) x_i x_j + \sum_{i,j=1}^n \partial_{\eta_i \eta_j}^2 Q(\zeta) y_i y_j \\ &\geq \frac{p+2\gamma}{2} [(p-2) |\xi|^{p-2} |\xi|^{-2} \langle \xi, x \rangle^2 + |x|^2] \\ &\quad + \frac{(q+\gamma(2-q))}{2} |\eta|^{q-2} [(q-2) |\eta|^{-2} \langle \eta, y \rangle^2 + |y|^2] \\ &\geq \frac{1}{2} [(p-1) |\xi|^{p-2} |x|^2 + (q-1) |\eta|^{q-2} |y|^2] \\ &\geq \frac{1}{2} [(p-1) |\xi|^{p-2} |x|^2 + \frac{|\xi|^{2-p}}{p-1} |y|^2] \\ &\geq \frac{1}{2} (\tau |x|^2 + \tau^{-1} |y|^2), \end{aligned}$$

with $\tau = (p-1) |\xi|^{p-2}$. Here in the second inequality we have used the facts that $p+2\gamma \geq 1$ and $q+\gamma(2-q) \geq 1$ and in the third inequality we have used the identity $q-1 = (p-1)^{-1}$ and the fact that $|\eta|^{q-2} \geq |\xi|^{p-2}$ in R_2 . \square

The Bellman function Q fails to be of class C^2 in all of $\mathbb{R}^m \times \mathbb{R}^n$ because the second derivatives are discontinuous on $|\xi|^p = |\eta|^q$. Since, for our purposes, we need to work with a Bellman function of class C^2 everywhere, we must replace the function Q by a regularised version that retains its essential properties.

To regularise Q we apply the standard technique of convolving with a mollifier. Let \mathcal{B}_1 be the open ball in \mathbb{R}^{m+n} with radius of length 1, and set

$$\phi(\zeta) = c e^{-\frac{1}{1-|\zeta|^2}} \chi_{\mathcal{B}_1}(\zeta),$$

where c is the normalization constant chosen in such a way that

$$\int_{\mathbb{R}^{m+n}} \phi(\zeta) d\zeta = 1.$$

For each $\sigma > 0$ we introduce the mollifier on \mathbb{R}^{m+n}

$$\phi_\sigma(\zeta) = \frac{1}{\sigma^{m+n}} \phi\left(\frac{\zeta}{\sigma}\right),$$

and we define the regularized version of the Bellman function Q :

$$Q_\sigma(\zeta) = Q_\sigma(\xi, \eta) = \phi_\sigma \star Q(\zeta),$$

where \star denotes the convolution in \mathbb{R}^{m+n} . Since both Q and ϕ_σ are separately radial in ξ and η , for each $\sigma > 0$ there exists a function

$$\beta_\sigma : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$$

such that for all $(\xi, \eta) \in \mathbb{R}^m \times \mathbb{R}^n$

$$Q_\sigma(\xi, \eta) = \frac{1}{2} \beta_\sigma(|\xi|, |\eta|).$$

Proposition 6.3. *If $0 < \sigma < 1$, then $Q_\sigma \in C^\infty(\mathbb{R}^m \times \mathbb{R}^n)$. Moreover for every $(u, v) \in \mathbb{R}_+ \times \mathbb{R}_+$, the following assertions hold:*

- (i') $0 \leq \beta_\sigma(u, v) \leq (1 + \gamma) [(u + \sigma)^p + (v + \sigma)^q]$;
- (ii') *there exists a constant C such that for every $u, v > 0$*

$$\begin{aligned} 0 &\leq \partial_u \beta_\sigma(u, v) \leq C p \max \{ (u + \sigma)^{p-1}, v + \sigma \}, \\ 0 &\leq \partial_v \beta_\sigma(u, v) \leq C (v + \sigma)^{q-1}. \end{aligned}$$

(iii') *for all $\zeta = (\xi, \eta) \in (\mathbb{R} \times \mathbb{R}^d)$ there exists $\tau_\sigma = \tau_\sigma(|\xi|, |\eta|) > 0$ such that*

$$H_{Q_\sigma}(\zeta; \omega) \geq \frac{1}{2} \gamma (\tau_\sigma |\rho|^2 + \tau_\sigma^{-1} |\psi|^2)$$

whenever $\omega = (\rho, \psi) \in \mathbb{R} \times \mathbb{R}^d$;

Proof. The estimate of β_σ derives from its definition and the properties of the Bellman function Q . Indeed, setting $u = |\xi|, v = |\eta|$ for some $(\xi, \eta) \in \mathbb{R} \times \mathbb{R}^d$,

$$\beta_\sigma(u, v) = \beta_\sigma(|\xi|, |\eta|) = 2Q_\sigma(\xi, \eta) \geq 0,$$

and

$$\begin{aligned}
\beta_\sigma(u, v) &= 2\phi_\sigma \star Q(\xi, \eta) = \\
&= 2 \int_{\mathbb{R}^{m+n}} \phi_\sigma(\xi', \eta') Q(\xi - \xi', \eta - \eta') d\xi' d\eta' \leq \\
&\leq (1 + \gamma) \int_{\mathbb{R}^{m+n}} \phi_\sigma(\xi', \eta') (|\xi - \xi'|^p + |\eta - \eta'|^q) d\xi' d\eta'.
\end{aligned}$$

Since $|\xi'| < \sigma$ and $|\eta'| < \sigma$ in the support of ϕ_σ , it follows that

$$\begin{aligned}
\beta_\sigma(u, v) &\leq (1 + \gamma) [(|\xi| + \sigma)^p + (|\eta| + \sigma)^q] \int_{\mathbb{R}^{m+n}} \phi_\sigma(\xi', \eta') d\xi' d\eta' \\
&= (1 + \gamma) [(u + \sigma)^p + (v + \sigma)^q].
\end{aligned}$$

Next we prove (ii'). With the change of variables $u = |\xi|$, $v = |\eta|$ we get

$$\partial_u \beta_\sigma(u, v) = \partial_{\xi_1} Q_\sigma(\xi, \eta) \frac{\xi_1}{|\xi|}, \quad \partial_v \beta_\sigma(u, v) = \partial_{\eta_1} Q_\sigma(\xi, \eta) \frac{\eta_1}{|\eta|}.$$

Therefore it suffices to show that $\partial_{\xi_1} Q_\sigma(\xi, \eta) \geq 0$ for $\xi_1 > 0$, $\partial_{\eta_1} Q_\sigma(\xi, \eta) \geq 0$ for $\eta_1 > 0$ and appropriate upper estimates for $|\partial_{\xi_1} Q_\sigma(\xi, \eta)|$ and $|\partial_{\eta_1} Q_\sigma(\xi, \eta)|$. We prove only the estimates of $\partial_{\xi_1} Q$, since the proof of those of $\partial_{\eta_1} Q_\sigma$ are similar. Write $\xi = (\xi_1, \hat{\xi})$ where $\hat{\xi} = (\xi_2, \dots, \xi_n)$ and set, for $t \in \mathbb{R}$, $\hat{\xi}, \hat{\xi}', \eta, \eta'$ fixed

$$f(t) = \partial_{\xi_1} Q(t, \hat{\xi} - \hat{\xi}', \eta - \eta'), \quad g(t) = \phi_\sigma(t, \hat{\xi}', \eta').$$

The function f is odd and nonnegative, g is even, nonnegative and decreasing on $[0, \infty)$. Thus, for $t > 0$

$$f \star g(t) = \int_0^t f(s) [g(t-s) - g(t+s)] ds + \int_t^\infty f(s) [g(s-t) - g(s+t)] ds \geq 0.$$

Hence, for $\xi_1 > 0$

$$\partial_{\xi_1} Q_\sigma(\xi, \eta) = \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}^n} f \star g(\xi_1) d\hat{\xi} d\eta \geq 0.$$

This proves that $\partial_u \beta_\sigma(u, v) \geq 0$.

To prove the upper estimate of $\partial_u \beta_\sigma(u, v)$ we observe that

$$\begin{aligned}
\partial_u \beta_\sigma(u, v) &= 2|\partial_{\xi_1} Q_\sigma(\xi, \eta)| \\
&= 2 \int_{\mathbb{R}^{m+n}} \phi_\sigma(\xi', \eta') |\partial_{\xi_1} Q(\xi - \xi', \eta - \eta')| d\xi' d\eta' \\
&\leq C \int_{\mathbb{R}^{m+n}} \phi_\sigma(\xi', \eta') \max \{|\xi - \xi'|^{p-1}, |\eta - \eta'|^q\} d\xi' d\eta' \leq \\
&\leq C \max \{(u + \sigma)^{p-1}, v + \sigma\} \int_{\mathbb{R}^{m+n}} \phi_\sigma(\xi', \eta') d\xi' d\eta' \leq \\
&\leq C \max \{(u + \sigma)^{p-1}, v + \sigma\},
\end{aligned}$$

where again we have used the fact that ϕ is supported in \mathcal{B}_1 , and that $p \geq 2$.

The proof of the inequalities for $\partial_v \beta_\sigma(u, v)$ is analogous.

Finally, we prove (iii'). Since the second order derivatives of Q are locally integrable

$$H_{Q_\sigma}(\zeta; z) = \int_{\mathbb{R}^{m+n}} H_Q(\zeta - \zeta'; z) \phi_\sigma(\zeta') d\zeta',$$

is well defined for each $\zeta = (\xi, \eta)$ and $z = (x, y)$ in $\mathbb{R}^m \times \mathbb{R}^n$. Therefore, by Proposition 6.2, there exists $\tau = \tau(|\xi - \xi'|, |\eta - \eta'|) > 0$ such that

$$\begin{aligned} H_{Q_\sigma}(\zeta; z) &\geq \frac{1}{2} \gamma \int_{\mathbb{R}^{m+n}} (\tau|x|^2 + \tau^{-1}|y|^2) \phi_\sigma(\zeta') d\zeta' = \\ &= \frac{1}{2} \gamma ((\tau \star \phi_\sigma)(\zeta)|x|^2 + (\tau^{-1} \star \phi_\sigma)(\zeta)|y|^2). \end{aligned}$$

By Hölder's inequality

$$\begin{aligned} (\tau \star \phi_\sigma)(\zeta) (\tau^{-1} \star \phi_\sigma)(\zeta) &= \\ &= \int_{\mathbb{R}^{m+n}} \tau(\zeta') \phi_\sigma(\zeta - \zeta') d\zeta' \int_{\mathbb{R}^{m+n}} \tau^{-1}(\zeta') \phi_\sigma(\zeta - \zeta') d\zeta' \\ &\geq \left(\int_{\mathbb{R}^{m+n}} \sqrt{\tau(\zeta') \phi_\sigma(\zeta - \zeta')} \sqrt{\tau^{-1}(\zeta') \phi_\sigma(\zeta - \zeta')} d\zeta' \right)^2 = 1. \end{aligned}$$

Thus

$$(\tau^{-1} \star \phi_\sigma)(\zeta) \geq (\tau \star \phi_\sigma)^{-1}(\zeta).$$

Hence, if we define

$$\tau_\sigma = \tau_\sigma(|\zeta|) = (\tau \star \phi_\sigma)(\zeta),$$

we obtain the desired estimate. \square

In the last part of this section we define the Bellman function on differential forms on \mathbb{R}_+^d and we prove a technical result that will be used in the proof of the Bilinear embedding Theorem.

For each $s = 1, \dots, d$ we set $d_s = \dim(\Lambda^s(\mathbb{R}^d)) = \binom{d}{s}$ and identify $\Lambda^s(\mathbb{R}^d)$ with \mathbb{R}^{d_s} via the map $\xi \mapsto (\xi_I)$ that associates to a s -form the vector of its components, in some fixed order (for instance the lexicographic order on the set of indices \mathcal{I}_s). Define the function $Q_\sigma : \Lambda^r(\mathbb{R}^d) \times \Lambda^{r+1}(\mathbb{R}^d) \rightarrow [0, \infty)$, by

$$Q_\sigma(\xi, \theta) = \frac{1}{2} \beta_\sigma(|\xi|, |\theta|)$$

If $\zeta = (\xi, \theta) \in C^\infty(\Lambda^r(\mathbb{R}_+^d \times \mathbb{R}_+)) \times C^\infty(\Lambda^{r+1}(\mathbb{R}_+^d \times \mathbb{R}_+))$, for each $i = 1, \dots, d+1$ denote by $\delta_i \zeta$ the vector in $\mathbb{R}^{d_r} \times \mathbb{R}^{d_{r+1}}$, whose components are

$$\begin{aligned} \delta_i \zeta_I &= \delta_i \xi_I & I \in \mathcal{I}_r \\ \delta_i \zeta_J &= \delta_i \theta_J & J \in \mathcal{I}_{r+1}. \end{aligned}$$

Here, as before, δ_i denotes the Laguerre derivative $\sqrt{x_i} \partial_i$ for $i = 1, \dots, d$, while $\delta_{d+1} = \partial_t$ is the classical derivative.

Define the operator M_α acting on r -forms by

$$M_\alpha \omega(x) = \sum_{I \in \mathcal{I}_r} M_{\alpha, I} \omega_I(x) dx_I,$$

where $M_{\alpha, I}$ is the operator of multiplication defined in (2.6).

Remark 6.4. If $\alpha \in [-1/2, \infty)^d$, then by (2.7)

$$\langle M_\alpha \omega(x), \omega(x) \rangle = \sum_{I \in \mathcal{I}_r} \omega_I(x) M_{\alpha, I} \omega_I(x) \geq \frac{r}{2} |\omega(x)|^2.$$

Lemma 6.5. *For every smooth r -form ω*

$$\sum_I \omega_I \mathcal{L}_\alpha \omega_I = \langle \mathbb{L}_\alpha \omega, \omega \rangle - \langle M_\alpha \omega, \omega \rangle.$$

Proof. A straightforward application of the identity $\mathcal{L}_\alpha = \mathcal{L}_{\alpha, I} - M_{\alpha, I}$ (see (2.5)), shows that

$$\begin{aligned} \sum_I \omega_I \mathcal{L}_\alpha \omega_I &= \sum_I \omega_I \mathcal{L}_{\alpha, I} \omega_I - \sum_I \omega_I M_{\alpha, I} \omega_I \\ &= \langle \mathbb{L}_\alpha \omega, \omega \rangle_{\wedge^r} - \langle M_\alpha \omega, \omega \rangle_{\wedge^r}. \end{aligned}$$

□

Define the differential operators on $\mathbb{R}_+^d \times \mathbb{R}_+$

$$\mathcal{D}_\alpha = \mathcal{L}_\alpha - \partial_{tt}^2 \quad \text{and} \quad \mathbb{D}_\alpha = \mathbb{L}_\alpha - \partial_{tt}^2.$$

Lemma 6.6. *Suppose that $\zeta = (\xi, \theta) \in C^\infty(\Lambda^r(\mathbb{R}_+^d \times \mathbb{R}_+)) \times C^\infty(\Lambda^{r+1}(\mathbb{R}_+^d \times \mathbb{R}_+))$. Then for every $\zeta \in \mathbb{R}_+^d \times \mathbb{R}_+$ and $\sigma \in (0, 1)$ we have*

$$\begin{aligned} -\mathcal{D}_\alpha Q_\sigma(\zeta) &= \sum_{i=1}^{d+1} H_{Q_\sigma}(\zeta; \delta_i \zeta) + \frac{\partial_1 \beta_\sigma(|\xi|, |\theta|)}{2|\xi|} \langle (M_\alpha - \mathbb{D}_\alpha) \xi, \xi \rangle + \\ &\quad + \frac{\partial_2 \beta_\sigma(|\xi|, |\theta|)}{2|\theta|} \langle (M_\alpha - \mathbb{D}_\alpha) \theta, \theta \rangle. \end{aligned}$$

Proof. First note that

$$(6.2) \quad -\mathcal{D}_\alpha Q_\sigma(\zeta) = \sum_{i=1}^d (\delta_{ii}^2 Q_\sigma(\zeta) + \psi_i \delta_i Q_\sigma(\zeta)) + \partial_{tt}^2 Q_\sigma(\zeta).$$

Since for $i = 1, \dots, d+1$

$$\begin{aligned} \delta_i Q_\sigma(\zeta) &= \sum_{I \in \mathcal{I}_r} \partial_{\xi_I} Q_\sigma(\zeta) \delta_i \xi_I + \sum_{J \in \mathcal{I}_{r+1}} \partial_{\theta_J} Q_\sigma(\zeta) \delta_i \theta_J, \\ \delta_{ii}^2 Q_\sigma(\zeta) &= \sum_{I_1 \in \mathcal{I}_r} \sum_{I_2 \in \mathcal{I}_r} \partial_{\xi_{I_1} \xi_{I_2}}^2 Q_\sigma(\zeta) \delta_i \xi_{I_1} \delta_i \xi_{I_2} + \sum_{I \in \mathcal{I}_r} \partial_{\xi_I} Q_\sigma(\zeta) \delta_{ii}^2 \xi_I + \\ &\quad + 2 \sum_{I \in \mathcal{I}_r} \sum_{J \in \mathcal{I}_{r+1}} \partial_{\xi_I \theta_J}^2 Q_\sigma(\zeta) \delta_i \xi_I \delta_i \theta_J + \\ &\quad + \sum_{J_1 \in \mathcal{I}_{r+1}} \sum_{J_2 \in \mathcal{I}_{r+1}} \partial_{\theta_{J_1} \theta_{J_2}}^2 Q_\sigma(\zeta) \delta_i \theta_{J_1} \delta_i \theta_{J_2} + \sum_{J \in \mathcal{I}_{r+1}} \partial_{\theta_J} Q_\sigma(\zeta) \delta_{ii}^2 \theta_J \\ &= H_{Q_\sigma}(\zeta; \delta_i \zeta) + \sum_{I \in \mathcal{I}_r} \partial_{\xi_I} Q_\sigma(\zeta) \delta_{ii}^2 \xi_I + \sum_{J \in \mathcal{I}_{r+1}} \partial_{\theta_J} Q_\sigma(\zeta) \delta_{ii}^2 \theta_J, \end{aligned}$$

the identity (6.2) can be rewritten as

$$-\mathcal{D}_\alpha Q_\sigma(\zeta) = \sum_{i=1}^{d+1} H_{Q_\sigma}(\zeta; \delta_i \zeta) - \sum_{I \in \mathcal{I}_r} \partial_{\xi_I} Q_\sigma(\zeta) \mathcal{D}_\alpha \xi_I - \sum_{J \in \mathcal{I}_{r+1}} \partial_{\theta_J} Q_\sigma(\zeta) \mathcal{D}_\alpha \theta_J.$$

Therefore, to conclude the proof, we only need to prove that

$$-\sum_{I \in \mathcal{I}_r} \partial_{\xi_I} Q_\sigma(\zeta) \mathcal{D}_\alpha \xi_I = \frac{\partial_1 \beta_\sigma(|\xi|, |\theta|)}{2|\xi|} \langle (M_\alpha - \mathbb{D}_\alpha) \xi, \xi \rangle,$$

and

$$(6.3) \quad - \sum_{J \in \mathcal{I}_{r+1}} \partial_{\theta_J} Q_\sigma(\zeta) \mathcal{D}_\alpha \theta_J = \frac{\partial_2 \beta_\sigma(|\xi|, |\theta|)}{2|\theta|} \langle (M_\alpha - \mathbb{D}_\alpha) \theta, \theta \rangle.$$

Indeed, since $Q_\sigma(\zeta) = \beta_\sigma(|\xi|, |\theta|)/2$, we have that

$$\partial_{\xi_I} Q_\sigma(\zeta) = \frac{\partial_1 \beta_\sigma(|\xi|, |\theta|)}{2|\xi|} \xi_I, \quad \partial_{\theta_J} Q_\sigma(\zeta) = \frac{\partial_2 \beta_\sigma(|\xi|, |\theta|)}{2|\theta|} \theta_J.$$

Thus

$$\begin{aligned} - \sum_{I \in \mathcal{I}_r} \partial_{\xi_I} Q_\sigma(\zeta) \mathcal{D}_\alpha \xi_I &= - \frac{\partial_1 \beta_\sigma(|\xi|, |\theta|)}{2|\xi|} \sum_{I \in \mathcal{I}_r} \xi_I \mathcal{D}_\alpha \xi_I \\ &= - \frac{\partial_1 \beta_\sigma(|\xi|, |\theta|)}{2|\xi|} \left(\sum_{I \in \mathcal{I}_r} \xi_I \mathcal{L}_\alpha \xi_I - \sum_{I \in \mathcal{I}_r} \xi_I \partial_{tt}^2 \xi_I \right) \\ &= - \frac{\partial_1 \beta_\sigma(|\xi|, |\theta|)}{2|\xi|} (\langle \mathbb{D}_\alpha \xi, \xi \rangle - \langle M_\alpha \xi, \xi \rangle), \end{aligned}$$

where in the last identity we have used Lemma 6.5. The identity (6.3) is proved similarly. \square

7. PROOF OF THE BILINEAR EMBEDDING THEOREM

In this section we assume that $\alpha \in [-1/2, \infty)^d$, $p \geq 2$, $q = p/(p-1)$ and $\gamma_p = q(q-1)/8$. Define the function

$$r(x) = r(x_1, \dots, x_d) = \sqrt{x_1 + \dots + x_d} \quad x \in \mathbb{R}_+^d.$$

We think of the function $r(x)$ as representing the distance of x from the origin of \mathbb{R}_+^d .

Choose a non-increasing cut-off function $\Theta \in C_c^\infty([0, \infty))$ such that $0 \leq \Theta \leq 1$, $\Theta(x) = 1$ if $x \in [0, 1]$ and $\Theta(x) = 0$ if $x \in [2, \infty)$. For $\ell > 0$ define

$$F_\ell(x) = \Theta \left(\frac{r(x)^2}{\ell^2} \right) = \Theta \left(\frac{x_1 + \dots + x_d}{\ell^2} \right)$$

Then $\text{supp } F_\ell \subseteq \overline{B_{\sqrt{2}\ell}(o)} = \{x \in \mathbb{R}_+^d : r(x) \leq \sqrt{2}\ell\}$.

Lemma 7.1. *There exists a constant $C = C(d, |\alpha|, \Theta) \geq 0$ such that*

$$|\delta F_\ell(x)| \leq C/\ell \quad \text{and} \quad \mathcal{L}_\alpha F_\ell(x) \leq C \quad \forall x \in \mathbb{R}_+^d \quad \forall \ell \geq 1.$$

Proof. Since $\mathcal{L}_\alpha r(x) = -\sum_{i=1}^d (\delta_{ii}^2 r(x) - \psi_i(x_i) \delta_i r(x))$, and

$$\delta_i r(x) = \frac{\sqrt{x_i}}{2r(x)}, \quad \delta_{ii}^2 r(x) = \frac{r^2(x) - x_i}{2r(x)}, \quad \psi_i(x_i) = \left(\frac{\alpha_i + 1/2}{\sqrt{x_i}} - \sqrt{x_i} \right),$$

a straightforward computation shows that $|\delta r(x)| \leq 1$ and, if $\ell \leq r \leq \sqrt{2}\ell$,

$$(7.1) \quad \frac{r \mathcal{L}_\alpha r}{\ell^2} = \frac{r}{\ell^2} \left[\frac{d + |\alpha|}{2r} - \left(\frac{1}{4r} + \frac{r}{2} \right) \right] \geq -\frac{5}{4}.$$

Thus

$$|\delta F_\ell(x)| \leq 2\|\Theta'\|_\infty \frac{r(x)}{\ell^2} |\delta r(x)| \leq C,$$

since $\ell \leq r \leq \sqrt{2}\ell$ on the support of Θ' .

Next observe that, setting $g = r^2/\ell^2$,

$$\begin{aligned}\mathcal{L}_\alpha(\Theta \circ g) &= -(\Theta'' \circ g)|\delta g|^2 + (\Theta' \circ g)\mathcal{L}_\alpha g \\ &= -\Theta''(r^2/\ell^2)\frac{4r^2|\delta r|^2}{\ell^4} + \Theta'(r^2/\ell^2)\left[\frac{2r\mathcal{L}_\alpha r}{\ell^2} - \frac{2|\delta r|^2}{\ell^2}\right].\end{aligned}$$

The desired conclusion follows, since Θ' and Θ'' are bounded, $\ell \leq r \leq \sqrt{2}\ell$ on the support of Θ' , $|\delta r| \leq 1$, $\Theta' \leq 0$ and $-r\mathcal{L}_\alpha r/\ell^2 \leq C$ by (7.1). \square

For all $s, \ell > 0$ we define

$$\mathcal{B}_{s,\ell} = \overline{B_{2\ell}(o)} \times [s^{-1}, s],$$

where

$$\overline{B_{2\ell}(o)} = \{x \in \mathbb{R}^d : r(x) \leq 2\ell\}$$

is the closed ball centered at the origin with radius 2ℓ with respect to the distance $r(x)$; moreover, for $\varepsilon > 0$ fixed, $\omega \in \mathcal{P}(\Lambda^r(\mathbb{R}_+^d))$ and $\eta \in \mathcal{P}(\Lambda^{r+1}(\mathbb{R}_+^d))$ set

$$\sigma_{s,\ell} = \varepsilon \inf_{(x,t) \in \mathcal{B}_{s,\ell}} \min\{P_t^\alpha |\omega|(x), P_t^\alpha |\eta|(x)\}.$$

Finally, define the function $b_{s,\ell} : \mathbb{R}_+^d \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by setting

$$b_{s,\ell}(x, t) = Q_{\sigma_{s,\ell}}(\mathbb{P}_t^{\alpha,\rho}\omega(x), \mathbb{P}_t^{\alpha,\rho}\eta(x)).$$

As in [11, 4] the main step of the proof of the bilinear embedding theorem consists in estimating an integral of $\mathbb{D}_\alpha b_{s,\ell}$. We begin with the estimate from below.

Lemma 7.2. *For every form $\phi \in L^2(\Lambda^m(\mathbb{R}_+^d), \mu_\alpha)$ and $\rho \leq m/2$*

$$\langle (M_\alpha - \mathbb{D}_\alpha)\mathbb{P}_t^{\alpha,\rho}\phi(x), \mathbb{P}_t^{\alpha,\rho}\phi(x) \rangle \geq 0.$$

Proof. Since $\mathbb{P}_t^{\alpha,\rho} = e^{-t(\mathbb{L}^\alpha - \rho I)^{1/2}}$ and $\mathbb{D}_\alpha = \mathbb{L}_\alpha - \partial_{tt}^2$, we have that $\mathbb{D}_\alpha \mathbb{P}_t^{\alpha,\rho}\phi(x) = \rho \mathbb{P}_t^{\alpha,\rho}\phi(x)$. Thus, since $\alpha \in [-1/2, \infty)^d$, by Remark 6.4,

$$\begin{aligned}\langle (M_\alpha - \mathbb{D}_\alpha)\mathbb{P}_t^{\alpha,\rho}\phi(x), \mathbb{P}_t^{\alpha,\rho}\phi(x) \rangle &= \langle M_\alpha \mathbb{P}_t^{\alpha,\rho}\phi(x), \mathbb{P}_t^{\alpha,\rho}\phi(x) \rangle - \rho |\mathbb{P}_t^{\alpha,\rho}\phi(x)|^2 \\ &\geq \left(\frac{m}{2} - \rho\right) |\mathbb{P}_t^{\alpha,\rho}\phi(x)|^2.\end{aligned}$$

\square

Proposition 7.3. *For all $(x, t) \in \mathbb{R}_+^d \times \mathbb{R}_+$*

$$-\mathcal{D}_\alpha b_{s,\ell}(x, t) \geq \gamma_p |\overline{\nabla} \mathbb{P}_t^{\alpha,\rho}\omega(x)| |\overline{\nabla} \mathbb{P}_t^{\alpha,\rho}\eta(x)|.$$

Proof. Set, for the sake of brevity, $\xi = \mathbb{P}_t^{\alpha,\rho}\omega(x)$, $\theta = \mathbb{P}_t^{\alpha,\rho}\eta(x)$ and $\zeta = (\xi, \theta)$. Then, by applying Lemma 6.6 we obtain that

$$\begin{aligned}-\mathcal{D}_\alpha b_{s,\ell}(x, t) &= \sum_{i=1}^{d+1} H_{Q_{\sigma_{s,\ell}}}(\zeta; \delta_i \zeta) \\ &\quad + \frac{\partial_1 \beta_\epsilon(|\xi|, |\theta|)}{2|\xi|} \langle (M_\alpha - \mathbb{D}_\alpha)\xi, \xi \rangle \\ &\quad + \frac{\partial_2 \beta_\epsilon(|\xi|, |\theta|)}{2|\theta|} \langle (M_\alpha - \mathbb{D}_\alpha)\theta, \theta \rangle.\end{aligned}$$

Since, by Proposition 6.3 (ii') the partial derivatives of β_ϵ are nonnegative, by Lemma 7.2 and Proposition 6.3 (iii')

$$\begin{aligned}
-\mathcal{D}_\alpha b_{s,\ell}(x,t) &\geq \sum_{i=1}^{d+1} H_{Q_{\sigma_{s,\ell}}}(\zeta; \delta_i \zeta) \\
&\geq \frac{\gamma_p}{2} \left(\tau_{\sigma_{s,\ell}} \sum_{i=1}^{d+1} |\delta_i \xi|^2 + \tau_{\sigma_{s,\ell}}^{-1} \sum_{i=1}^{d+1} |\delta_i \theta|^2 \right) \\
&\geq \gamma_p |\bar{\nabla} \xi|^2 |\bar{\nabla} \theta|^2.
\end{aligned}$$

□

To estimate $-\mathcal{D}_\alpha b_{s,\ell}$ from above we need the following two lemmas.

Lemma 7.4. *Suppose that $\rho \leq r/2$. Then for every $(x, t) \in \mathcal{B}_{s,\ell}$*

$$b_{s,\ell}(x, t) \leq C \frac{1+\gamma_p}{2} (1+\varepsilon)^p [P_t^\alpha |\omega|^p(x) + P_t^\alpha |\eta|^q(x)].$$

Moreover, there exists a constant $C = C(p, \varepsilon)$ such that for each $(x, t) \in \mathcal{B}_{s,\ell}$

$$\begin{aligned}
|\partial_t b_{s,\ell}(x, t)| &\leq C \max \{ (P_t^\alpha |\omega|(x))^{p-1}, P_t^\alpha |\eta|(x) \} |\partial_t \mathbb{P}_t^{\alpha, \rho} \omega(x)| \\
&\quad + C (P_t^\alpha |\eta|(x))^{q-1} |\partial_t \mathbb{P}_t^{\alpha, \rho} \eta(x)|.
\end{aligned}$$

Proof. By Proposition 6.3

$$\begin{aligned}
b_{s,\ell}(x, t) &= Q_{\sigma_{s,\ell}}(\mathbb{P}_t^{\alpha, \rho} \omega(x), \mathbb{P}_t^{\alpha, \rho} \eta(x)) \\
&= \frac{1}{2} \beta_{\sigma_{s,\ell}}(|\mathbb{P}_t^{\alpha, \rho} \omega(x)|, |\mathbb{P}_t^{\alpha, \rho} \eta(x)|) \\
&\leq \frac{1+\gamma_p}{2} [(|\mathbb{P}_t^{\alpha, \rho} \omega(x)| + \sigma_{s,\ell})^p + (|\mathbb{P}_t^{\alpha, \rho} \eta(x)| + \sigma_{s,\ell})^q].
\end{aligned}$$

Since $\rho \leq r/2$ and $\alpha \in [-1/2, \infty)^d$, we have that $|\mathbb{P}_t^{\alpha, \rho} \omega(x)| \leq P_t^\alpha |\omega|(x)$ and $|\mathbb{P}_t^{\alpha, \rho} \eta(x)| \leq P_t^\alpha |\eta|(x)$ by Proposition 3.17. Thus, since

$$\sigma_{s,\ell} \leq \varepsilon \inf_{\mathcal{B}_{s,\ell}} \min \{ P_t^\alpha |\omega|(x), P_t^\alpha |\eta|(x) \}$$

by definition, and $p \geq q$,

$$b_{s,\ell}(x, t) \leq \frac{1+\gamma_p}{2} (1+\varepsilon)^p (P_t^\alpha |\omega|^p(x) + P_t^\alpha |\eta|^q(x)).$$

To prove the second part of the statement, we apply Propositions 3.17 and 6.3 once more

$$\begin{aligned}
|\partial_t b_{s,\ell}(x, t)| &\leq \frac{1}{2} [\partial_1 \beta_{\sigma_{s,\ell}}(|\mathbb{P}_t^{\alpha, \rho} \omega(x)|, |\mathbb{P}_t^{\alpha, \rho} \eta(x)|) |\partial_t \mathbb{P}_t^{\alpha, \rho} \omega(x)| \\
&\quad + \partial_2 \beta_{\sigma_{s,\ell}}(|\mathbb{P}_t^{\alpha, \rho} \omega(x)|, |\mathbb{P}_t^{\alpha, \rho} \eta(x)|) |\partial_t \mathbb{P}_t^{\alpha, \rho} \eta(x)|] \\
&\leq C \max \{ (P_t^\alpha |\omega|(x))^{p-1}, P_t^\alpha |\eta|(x) \} |\partial_t \mathbb{P}_t^{\alpha, \rho} \omega(x)| \\
&\quad + C (P_t^\alpha |\eta|(x))^{q-1} |\partial_t \mathbb{P}_t^{\alpha, \rho} \eta(x)|.
\end{aligned}$$

□

In the following we set

$$R(x, t) = \frac{1+\gamma_p}{2} (1+\varepsilon)^p [P_t^\alpha |\omega|^p(x) + P_t^\alpha |\eta|^q(x)].$$

Lemma 7.5. *There exists a constant $C = C(p, \varepsilon, \omega, \eta)$ such that*

$$\int_{\mathbb{R}_+^d} |\delta R(x, t)| d\mu_\alpha(x) \leq C \quad \forall t > 0.$$

Proof. Since $\delta P_t^\alpha = \mathbb{P}_t^{\alpha, 0} \delta$, by Proposition 3.17

$$|\delta P_t^\alpha |\omega|^p(x)| = |\mathbb{P}_t^{\alpha, 0} \delta |\omega|^p|(x) \leq P_t^\alpha |\delta |\omega|^p|(x)$$

and, similarly, $|\delta P_t^\alpha |\eta|^q(x)| \leq P_t^\alpha |\delta |\eta|^q|(x)$. Next, observe that, since the coefficients of the forms ω and η are products of polynomials by square roots of the coordinates, and $p \geq q > 1$, the functions $|\omega|^p$ and $|\eta|^q$ are differentiable and

$$|\delta |\omega|^p| \leq p |\omega|^{p-1} |\delta \omega|, \quad |\delta |\eta|^q| \leq q |\eta|^{q-1} |\delta \eta|.$$

Hence the functions $|\delta |\omega|^p|$ and $|\delta |\eta|^q|$ are in $L^1(\mathbb{R}_+^d, \mu_\alpha)$ and, since P_t^α is a contraction on $L^1(\mathbb{R}_+^d, \mu_\alpha)$,

$$\int_{\mathbb{R}_+^d} |\delta R(x, t)| d\mu_\alpha(x) \leq C (\|\delta |\omega|^p\|_{L^1(\mu_\alpha)} + \|\delta |\eta|^q\|_{L^1(\mu_\alpha)}).$$

□

We are now ready to state and prove the estimate from above.

Proposition 7.6. *Suppose that $\rho \leq r/2$. Then*

$$\begin{aligned} \limsup_{s \rightarrow \infty} \limsup_{\ell \rightarrow \infty} \int_{s^{-1}}^s \int_{B_\ell(o)} -\mathcal{D}_\alpha b_{s,\ell}(x, t) d\mu_\alpha(x) t dt \\ \leq \frac{1 + \gamma_p}{2} (1 + \varepsilon)^p \left(\|\omega\|_{L^p(\mu_\alpha)}^p + \|\eta\|_{L^q(\mu_\alpha)}^q \right). \end{aligned}$$

Proof. We observe that $(\text{supp } F_\ell) \times [s^{-1}, s] \subseteq \mathcal{B}_{s,\ell}$. Since $-\mathcal{D}_\alpha b_{s,\ell} \geq 0$ by Proposition 7.3,

$$\int_{s^{-1}}^s \int_{B_\ell(o)} -\mathcal{D}_\alpha b_{s,\ell}(x, t) d\mu_\alpha(x) t dt \leq \int_{s^{-1}}^s \int_{\mathbb{R}_+^d} -\mathcal{D}_\alpha b_{s,\ell}(x, t) F_\ell(x) d\mu_\alpha(x) t dt.$$

Since $\mathcal{D}_\alpha = \mathcal{L}_\alpha - \partial_{tt}^2$, to complete the proof it is enough to show that

$$\begin{aligned} \limsup_{s \rightarrow \infty} \limsup_{\ell \rightarrow \infty} \int_{s^{-1}}^s \int_{\mathbb{R}_+^d} \partial_{tt}^2 b_{s,\ell}(x, t) F_\ell(x) d\mu_\alpha(x) t dt \\ \leq \frac{1 + \gamma_p}{2} (1 + \varepsilon)^p \left(\|\omega\|_{L^p(\mu_\alpha)}^p + \|\eta\|_{L^q(\mu_\alpha)}^q \right) \end{aligned} \quad (7.2)$$

and

$$\lim_{\ell \rightarrow \infty} \int_{s^{-1}}^s \int_{\mathbb{R}_+^d} \mathcal{L}_\alpha b_{s,\ell}(x, t) F_\ell(x) d\mu_\alpha(x) t dt = 0 \quad \text{for all } s > 0. \quad (7.3)$$

First we prove (7.2). Integrating by parts in the variable t we get

$$\begin{aligned} \int_{s^{-1}}^s \partial_{tt}^2 b_{s,\ell}(x, t) t dt &= t \partial_t b_{s,\ell}(x, t) \Big|_{s^{-1}}^s - \int_{s^{-1}}^s \partial_t b_{s,\ell}(x, t) dt \\ &= s \partial_t b_{s,\ell}(x, s) - s^{-1} \partial_t b_{s,\ell}(x, s^{-1}) + b_{s,\ell}(x, s^{-1}) - b_{s,\ell}(x, s). \end{aligned}$$

Since $b_{s,\ell}(x, s) \geq 0$, by Lemma 7.4 we have that

$$b_{s,\ell}(x, s^{-1}) - b_{s,\ell}(x, s) \leq b_{s,\ell}(x, s^{-1}) \frac{1 + \gamma_p}{2} (1 + \varepsilon)^p (P_{s^{-1}}^\alpha |\omega(x)|^p + P_{s^{-1}}^\alpha |\eta(x)|^q).$$

Thus we obtain the estimate

$$\begin{aligned} & \int_{s^{-1}}^s \int_{\mathbb{R}_+^d} \partial_{tt}^2 b_{s,\ell}(x, t) F_\ell(x) d\mu_\alpha(x) t dt \\ & \leq \frac{1 + \gamma_p}{2} (1 + \varepsilon)^p \int_{\mathbb{R}_+^d} (P_{s^{-1}}^\alpha |\omega(x)|^p + P_{s^{-1}}^\alpha |\eta(x)|^q) d\mu_\alpha(x) \\ & \quad + \int_{B_{2\ell}(o)} [s \partial_t b_{s,\ell}(x, s) - s^{-1} \partial_t b_{s,\ell}(x, s^{-1})] d\mu_\alpha(x) \\ & \leq \frac{1 + \gamma_p}{2} (1 + \varepsilon)^p \left(\|\omega\|_{L^p(\mu_\alpha)}^p + \|\eta\|_{L^q(\mu_\alpha)}^q \right) \\ & \quad + \|s \partial_t b_{s,\ell}(\cdot, s)\|_{L^1(B_{2\ell}(o), \mu_\alpha)} + \|s^{-1} \partial_t b_{s,\ell}(\cdot, s^{-1})\|_{L^1(B_{2\ell}(o), \mu_\alpha)} \end{aligned}$$

by the contractivity of the Poisson semigroup P_t^α on $L^r(\mathbb{R}_+^d, \mu_\alpha)$ for all $r \in [1, \infty)$. Therefore, to conclude the proof of (7.2), it is enough to show that

$$(7.4) \quad \limsup_{s \rightarrow 0} \sup_{\ell} \|s \partial_t b_{s,\ell}(\cdot, s)\|_{L^1(B_{2\ell}(o), \mu_\alpha)} = \limsup_{s \rightarrow \infty} \sup_{\ell} \|s \partial_t b_{s,\ell}(\cdot, s)\|_{L^1(B_{2\ell}(o), \mu_\alpha)} = 0.$$

Fix v such that $v(q-1) > 1$; then $v > 2$. By Lemma 7.4 and Hölder's inequality, to prove (7.4) it suffices to show that

$$(7.5) \quad \sup_{t > 0} \|(P_t^\alpha |\omega|)^{p-1} + P_t^\alpha |\eta| + (P_t^\alpha |\eta|)^{q-1}\|_{L^v(\mu_\alpha)} \leq C(p, \omega, \eta)$$

and

$$(7.6) \quad \|t \partial_t \mathbb{P}_t^{\alpha, \rho} \omega\|_{L^{v'}(\mu_\alpha)} + \|t \partial_t \mathbb{P}_t^{\alpha, \rho} \eta\|_{L^{v'}(\mu_\alpha)} \rightarrow 0 \quad \text{as } t \rightarrow 0, \infty.$$

Now, (7.5) follows from the fact that the Poisson semigroup P_t^α is a contraction on $L^r(\mathbb{R}_+^d, \mu_\alpha)$ for all $r \in [1, \infty)$, whereas the spectral theorem implies that (7.6) holds for the norms in $L^2(\mu_\alpha)$. Since $v' < 2$ and the measure μ_α is finite it holds also for the norms in $L^{v'}(\mu_\alpha)$. This concludes the proof of (7.2).

Next we prove (7.3). Integrating by parts twice we obtain

$$\int_{\mathbb{R}_+^d} \mathcal{L}_\alpha b_{s,\ell}(x, t) F_\ell(x) d\mu_\alpha(x) = \int_{\mathbb{R}_+^d} b_{s,\ell}(x, t) \mathcal{L}_\alpha F_\ell(x) d\mu_\alpha(x).$$

Note that in the integrations by parts the boundary terms vanish, since both $x \mapsto b_{s,\ell}(x, t)$ and F_ℓ are smooth up to the boundary of \mathbb{R}_+^d and F_ℓ has compact support. Thus, since $b_{s,\ell} \geq 0$ and $\text{supp}(\mathcal{L}_\alpha F_\ell) \subset B_{\sqrt{2}\ell}(o) \setminus B_\ell(o)$, by Lemma 7.1 we have that

$$\begin{aligned} \int_{\mathbb{R}_+^d} \mathcal{L}_\alpha b_{s,\ell}(x, t) F_\ell(x) d\mu_\alpha(x) & \leq C \int_{B_{\sqrt{2}\ell}(o) \setminus B_\ell(o)} b_{s,\ell}(x, t) d\mu_\alpha(x) \\ & \leq C \int_{B_{\sqrt{2}\ell}(o) \setminus B_\ell(o)} [P_t^\alpha |\omega|^p(x) + P_t^\alpha |\eta|^q(x)] d\mu_\alpha(x). \end{aligned}$$

Set

$$\Xi_\ell(t) = \int_{B_{\sqrt{2}\ell}(o) \setminus B_\ell(o)} [P_t^\alpha |\omega|^p(x) + P_t^\alpha |\eta|^q(x)] d\mu_\alpha(x).$$

Then $\lim_{\ell \rightarrow \infty} \Xi_\ell(t) = 0$ and $0 \leq \Xi_\ell(t) \leq \|\omega\|_{L^p(\mu_\alpha)}^p + \|\eta\|_{L^q(\mu_\alpha)}^q$. Hence, by the Lebesgue dominated convergence theorem,

$$\limsup_{\ell \rightarrow \infty} \int_{s^{-1}}^s \int_{\mathbb{R}_+^d} \mathcal{L}_\alpha b_{s,\ell}(x, t) F_\ell(x) d\mu_\alpha(x) t dt \leq 0.$$

Thus, to conclude the proof of (7.3) it remains only to prove that

$$(7.7) \quad \liminf_{\ell \rightarrow \infty} \int_{s^{-1}}^s \int_{\mathbb{R}_+^d} \mathcal{L}_\alpha b_{s,\ell}(x, t) F_\ell(x) d\mu_\alpha(x) t dt \geq 0.$$

By adding and subtracting to $b_{s,\ell}$ the function

$$R(x, t) = \frac{1 + \gamma_p}{2} (1 + \varepsilon)^p [P_t^\alpha |\omega|^p(x) + P_t^\alpha |\eta|^q(x)]$$

we may write the integral in left hand side of (7.7) as

$$\int_{s^{-1}}^s \int_{\mathbb{R}_+^d} \mathcal{L}_\alpha (b_{s,\ell} - R) F_\ell d\mu_\alpha t dt + \int_{s^{-1}}^s \int_{\mathbb{R}_+^d} \mathcal{L}_\alpha R F_\ell d\mu_\alpha t dt.$$

Therefore, to prove (7.7) it suffices to show that

$$(7.8) \quad \limsup_{\ell \rightarrow \infty} \int_{s^{-1}}^s \int_{\mathbb{R}_+^d} \mathcal{L}_\alpha (R - b_{s,\ell}) F_\ell d\mu_\alpha t dt \leq 0,$$

and

$$(7.9) \quad \lim_{\ell \rightarrow \infty} \int_{s^{-1}}^s \int_{\mathbb{R}_+^d} \mathcal{L}_\alpha R F_\ell d\mu_\alpha t dt = 0.$$

To prove (7.8) we proceed much as before. Since $R - b_{s,\ell} \geq 0$ by Lemma 7.4, integrating by parts twice and using Lemma 7.1 we obtain that

$$\begin{aligned} \int_{\mathbb{R}_+^d} \mathcal{L}_\alpha (R - b_{s,\ell}) F_\ell d\mu_\alpha &= \int_{\mathbb{R}_+^d} (R - b_{s,\ell}) \mathcal{L}_\alpha F_\ell d\mu_\alpha \\ &\leq C \int_{B_{\sqrt{2}\ell}(o) \setminus B_\ell(o)} (R - b_{s,\ell}) d\mu_\alpha \\ &\leq C \int_{B_{\sqrt{2}\ell}(o) \setminus B_\ell(o)} R d\mu_\alpha \\ &= C \Xi_\ell(t). \end{aligned}$$

Hence (7.8) follows by the Lebesgue dominated convergence theorem. It remains only to prove (7.9). By integrating by parts and using Lemmas 7.4 and 7.5, we obtain that

$$\begin{aligned} \left| \int_{s^{-1}}^s \int_{\mathbb{R}_+^d} \mathcal{L}_\alpha R F_\ell d\mu_\alpha t dt \right| &\leq \int_{s^{-1}}^s \int_{\mathbb{R}_+^d} |\delta R| |\delta F_\ell| d\mu_\alpha t dt \\ &\leq C/\ell. \end{aligned}$$

This concludes the proof of (7.9) and of the proposition. \square

Proof of the Bilinear embedding Theorem. First we prove the statement for $p \geq 2$. By Propositions 7.3 and 7.6 and passing to the limit as $\epsilon \rightarrow 0$ we obtain

$$(7.10) \quad \begin{aligned} & \gamma_p \int_0^\infty \int_{\mathbb{R}_+^d} |\bar{\nabla} \mathbb{P}_t^{\alpha, \rho} \omega(x)| |\bar{\nabla} \mathbb{P}_t^{\alpha, \rho} \eta(x)(x)| d\mu_\alpha(x) t dt \\ & \leq \frac{1 + \gamma_p}{2} \left(\|\omega\|_{L^p(\mu_\alpha)}^p + \|\eta\|_{L^q(\mu_\alpha)}^q \right). \end{aligned}$$

By replacing ω with $\lambda\omega$ and η with $\lambda^{-1}\eta$ in this inequality and minimizing for $\lambda > 0$ we get

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}_+^d} |\bar{\nabla} \mathbb{P}_t^{\alpha, \rho} \omega(x)| |\bar{\nabla} \mathbb{P}_t^{\alpha, \rho} \eta(x)(x)| d\mu_\alpha(x) t dt \\ & \leq \frac{1 + \gamma_p}{2\gamma_p} \left[\left(\frac{p}{q} \right)^{1/p} + \left(\frac{q}{p} \right)^{1/q} \right] \|\omega\|_{L^p(\mu_\alpha)} \|\eta\|_{L^q(\mu_\alpha)}. \end{aligned}$$

Since $\gamma_p = q(q-1)/8$, we have that

$$\begin{aligned} \frac{1 + \gamma_p}{2\gamma_p} \left[\left(\frac{q}{p} \right)^{1/q} + \left(\frac{p}{q} \right)^{1/p} \right] &= \frac{8 + q(q-1)}{2} (q-1)^{\frac{1}{q}-1} (p-1) \\ &< 5.7 (p-1). \end{aligned}$$

Since $p^* = \max\{p, q\} = p$ when $p > 2$, this proves the Bilinear embedding Theorem for $p \geq 2$. The proof in the case $1 < p < 2$ is similar: it suffices to replace the constants $\gamma_p = q(q-1)/8$ and $(1+\epsilon)^p$ in Lemma 7.4 and in Proposition 7.6 by $\gamma_q = p(p-1)/8$ and $(1+\epsilon)^q$, and in the proof of (7.4) to fix v such that $v(p-1) > 1$. Proceeding as before we obtain the result also for $1 < p < 2$. \square

8. SPECTRAL MULTIPLIERS

In Section 3.4 we have seen that the Hodge-Laguerre operator on r -forms has a self-adjoint extension \mathbb{L}_α on $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ with spectral resolution

$$\mathbb{L}_\alpha = \sum_{n \geq r} n \mathcal{P}_n^\alpha,$$

where, for each integer $n \geq r$, \mathcal{P}_n is the orthogonal projection onto the space spanned by the forms $\lambda_k^\alpha(x) = \sum_{I \in \mathcal{I}_r} \ell_{k,I}^\alpha dx_I$, $|k| = n$. Thus, by the spectral theorem, if $m = (m(n))_{n \geq r}$ is a bounded sequence in \mathbb{C} the operator

$$m(\mathbb{L}_\alpha) = \sum_{n \geq r} m(n) \mathcal{P}_n^\alpha$$

is bounded on $L^2(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ and $\|m(\mathbb{L}_\alpha)\|_{2 \rightarrow 2} = \sup_{n \geq r} |m(n)|$.

In this section we shall give a sufficient condition for the boundedness of $m(\mathbb{L}_\alpha)$ on $L^p(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ also for $p \neq 2$.

Before stating and proving the multiplier theorem for \mathbb{L}_α , we briefly recall the universal multiplier theorem for symmetric contraction semigroups of Carbonaro and Dragičević [5]. We denote by $H^\infty(S_\theta)$ the space of all functions that are bounded and holomorphic in the sector

$$S_\theta = \{z \in \mathbb{C} : |\arg z| < \theta\}.$$

By Fatou's theorem a function m in $H^\infty(S_\theta)$ has non tangential limits almost everywhere on the boundary of S_θ . We denote by m_\pm the boundary values of m on the edges of the sector, considered as functions on \mathbb{R}_+ , i.e.

$$m_\pm(\lambda) = m(\lambda e^{\pm i\theta}) \quad \forall \lambda \in \mathbb{R}_+.$$

For every $r > 0$ let $D_r m_\pm(\lambda) = m_\pm(r\lambda)$. For every $J > 0$ denote by $H^J(\mathbb{R})$ the usual L^2 -Sobolev space on \mathbb{R} . Fix a smooth function ψ with compact support in $[1/4, 4]$ such that $\psi = 1$ on $[1/2, 2]$.

Definition 8.1. We denote by $H^\infty(S_\theta; J)$ the space of all functions $m \in H^\infty(S_\theta)$ such that

$$\|m\|_{S_\theta; J} = \sup_{r>0} \|\psi D_r m_+\|_{H^J} + \sup_{r>0} \|\psi D_r m_-\|_{H^J} < \infty.$$

Then $H^\infty(S_\theta; J)$ does not depend on the choice of the function ψ and it is a Banach space with respect to the norm $\|\cdot\|_{S_\theta; J}$. For every $p \in [1, \infty]$ set

$$\phi_p^* = \arcsin \left| \frac{2}{p} - 1 \right|.$$

Theorem 8.2 (Carbonaro and Dragičević). *For every generator \mathcal{A} of a symmetric contraction semigroup on a σ -finite measure space (Ω, ν) , every $p \in (1, \infty)$, $J > 3/2$ and $m \in H^\infty(S_{\phi_p^*}; J)$, the operator $m(\mathcal{A})$ extends to a bounded operator on $L^p(\Omega, \nu)$, and there exists $C(p, J) > 0$ such that*

$$\|m(\mathcal{A})\|_{p-p} \leq C(p, J) \left(\|m\|_{H^\infty(S_{\phi_p^*}; J)} + |m(0)| \right).$$

Remark 8.3. It follows from the proof of Theorem 1 in [5] that if the operator \mathcal{A} is injective, then the term $|m(0)|$ can be omitted in the right hand side of the estimate of $\|m(\mathcal{A})\|_{p-p}$.

For each $a \geq 0$ define the translated sector

$$\tau_a S_{\phi_p^*} = \{z \in \mathbb{C} : \arg(z - a) < \phi_p^*\}.$$

and the space $H^\infty(\tau_a S_{\phi_p^*}; J)$ as the space of all functions m such that $z \mapsto \tau_a m(z) = m(z + a) \in H^\infty(S_{\phi_p^*}; J)$ endowed with the norm

$$\|m\|_{\tau_a S_{\phi_p^*}; J} = \|\tau_a m\|_{S_{\phi_p^*}; J}$$

Theorem 8.4. *Suppose that $\alpha \in (-1/2, \infty)^d$. If $1 \leq r \leq d$ and $m \in H^\infty(\tau_{r/2} S_{\phi_p^*}; J)$ for some $p \in (1, 2)$ and some $J > 3/2$, then the operator $m(\mathbb{L}_\alpha)$ is bounded on $L^q(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ for all $q \in [p, p']$ and*

$$\|m(\mathbb{L}_\alpha)\|_{q-q} \leq C(p, J, r) \|m\|_{\tau_{r/2} S_{\phi_p^*}; J}.$$

Proof. Suppose that $\omega = \sum_{I \in \mathcal{I}_r} \omega_I dx_I \in \mathcal{P}(\Lambda^r(\mathbb{R}_+^d))$. Then, by Proposition 3.6 and the spectral theorem

$$\begin{aligned} m(\mathbb{L}_\alpha) \omega &= \sum_{I \in \mathcal{I}_r} m(\mathcal{L}_{\alpha, I}) \omega_I dx_I \\ &= \sum_{I \in \mathcal{I}_r} \tau_{r/2} m(\mathcal{L}_{\alpha, I} - (r/2)I) \omega_I dx_I. \end{aligned}$$

By Corollary 2.5 the operators $\mathcal{L}_{\alpha,I} - (r/2)I$ generate semigroups of symmetric contractions on $L^q(\mathbb{R}_+^d, \mu_\alpha)$ for every $q \in [1, \infty]$. Since $\tau_{r/2}m \in H^\infty(S_{\phi_p}^*; J)$ the operators $m(\mathcal{L}_{\alpha,I}) = \tau_{r/2}m(\mathcal{L}_{\alpha,I} - (r/2)I)$ are bounded on $L^p(\mathbb{R}_+^d, \mu_\alpha)$ and

$$\|m(\mathcal{L}_{\alpha,I})\|_{p-p} \leq C(J, p) \|m\|_{\tau_{r/2}S_{\phi_p}^*; J}$$

by Theorem 8.2. To conclude that $m(\mathbb{L}_\alpha)$ is bounded on $L^p(\mathbb{R}_+^d, \mu_\alpha; \wedge^r)$ we apply a randomisation argument based on Rademacher's functions. We recall that the Rademacher functions are an orthonormal family of $\{-1, 1\}$ -valued functions $\{r_k : k \in \mathbb{N}\}$ in $L^2([0, 1], dt)$ such that if $F(t) = \sum_{n=0}^\infty a_n r_n(t)$ is a function in $L^2([0, 1], dt)$ then $F \in L^p([0, 1], dt)$ for every $p < \infty$ and

$$(8.1) \quad A_p \|F\|_p \leq \|F\|_2 = \left(\sum_{k=0}^\infty |a_k|^2 \right)^{1/2} \leq B_p \|F\|_p$$

for two positive constants A_p and B_p (see [16, Appendix C]). Let I_1, \dots, I_{d_r} , where $d_r = \#\mathcal{I}_r$, be an enumeration of the multiindices in \mathcal{I}_r . Then, by applying the second inequality in (8.1) to the function

$$F(t) = \sum_{k=1}^{d_r} m(\mathcal{L}_{\alpha, I_k}) \omega_{I_k}(x) r_k(t),$$

we obtain that

$$\begin{aligned} |m(\mathbb{L}_\alpha) \omega(x)| &= \left(\sum_{k=1}^{d_r} |m(\mathcal{L}_{\alpha, I_k}) \omega_{I_k}(x)|^2 \right)^{1/2} \\ &\leq B_p \left(\int_0^1 \left| \sum_{k=1}^{d_r} m(\mathcal{L}_{\alpha, I_k}) \omega_{I_k}(x) r_k(t) \right|^p dt \right)^{1/p}. \end{aligned}$$

Thus, since $|r_k(t)| = 1$ for every $t \in [0, 1]$, an application of Fubini's theorem and Minkowski's inequality yield

$$\begin{aligned} \|m(\mathbb{L}_\alpha) \omega\|_{L^p(\mu_\alpha)} &= \left(\int_{\mathbb{R}_+^d} |m(\mathbb{L}_\alpha) \omega(x)|^p d\mu_\alpha(x) \right)^{1/p} \\ &\leq B_p \left(\int_{\mathbb{R}_+^d} \int_0^1 \left| \sum_{k=1}^{d_r} m(\mathcal{L}_{\alpha, I_k}) \omega_{I_k}(x) r_k(t) \right|^p dt d\mu_\alpha(x) \right)^{1/p} \\ &\leq B_p \left(\int_{\mathbb{R}_+^d} \left(\sum_{k=1}^{d_r} |m(\mathcal{L}_{\alpha, I_k}) \omega_{I_k}(x)| \right)^p d\mu_\alpha(x) \right)^{1/p} \\ &\leq \sum_{k=1}^{d_r} \|m(\mathcal{L}_{\alpha, I_k}) \omega_{I_k}\|_{L^p(\mu_\alpha)} \\ &\leq \sum_{k=1}^{d_r} \|m(\mathcal{L}_{\alpha, I_k})\|_{p-p} \|\omega_{I_k}\|_{L^p(\mu_\alpha)} \\ &\leq C(p, J) d_r \|\omega\|_{L^p(\mu_\alpha)}. \end{aligned}$$

Hence $m(\mathbb{L}_\alpha)$ is bounded on $L^p(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$. Let $m^*(z) = \overline{m(\bar{z})}$. Then $m^* \in H^\infty(\tau_a S_{\phi_p^*}; J)$ and $\|m^*\|_{\tau_{r/2} S_{\phi_p^*}; J} = \|m\|_{\tau_{r/2} S_{\phi_p^*}; J}$. Then $m^*(\mathbb{L}_\alpha)$ is bounded on $L^p(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ and

$$\|m(\mathcal{L}_{\alpha, I})\|_{p \rightarrow p} \leq C(J, p) \|m\|_{\tau_{r/2} S_{\phi_p^*}; J}.$$

Since $m(\mathbb{L}_\alpha)^* = m^*(\mathbb{L}_\alpha)$ because \mathbb{L}_α is self-adjoint, the operator $m(\mathbb{L}_\alpha) = m^*(\mathbb{L}_\alpha)^*$ is also bounded on $L^{p'}(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$, $p' = p/(p-1)$ by duality. Thus it is bounded on $L^q(\mathbb{R}_+^d, \mu_\alpha; \Lambda^r)$ for all $q \in [p, p']$ by interpolation. This concludes the proof of the theorem. \square

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REFERENCES

- [1] P. Auscher, T. Coulhon, X. T. Duong, and S. Hofmann. Riesz transform on manifolds and heat kernel regularity. *Ann. Sci. École Norm. Sup. (4)*, 37(6):911–957, 2004.
- [2] D. Bakry. Étude des transformations de Riesz dans les variétés riemanniennes à courbure de Ricci minorée. In *Séminaire de Probabilités, XXI*, volume 1247 of *Lecture Notes in Math.*, pages 137–172. Springer, Berlin, 1987.
- [3] E. L. Bueler. The heat kernel weighted Hodge Laplacian on noncompact manifolds. *Trans. Amer. Math. Soc.*, 351(2):683–713, 1999.
- [4] A. Carbonaro and O. Dragičević. Bellman function and linear dimension-free estimates in a theorem of Bakry. *J. Funct. Anal.*, 265(7):1085–1104, 2013.
- [5] A. Carbonaro and O. Dragičević. Functional calculus for generators of symmetric contraction semigroups. arXiv:1308.1338, 2013.
- [6] G. Carron, T. Coulhon, and A. Hassell. Riesz transform and L^p -cohomology for manifolds with Euclidean ends. *Duke Math. J.*, 133(1):59–93, 2006.
- [7] R. R. Coifman, R. Rochberg, and G. Weiss. Applications of transference: the L^p version of von Neumann’s inequality and the Littlewood-Paley-Stein theory. In *Linear spaces and approximation (Proc. Conf., Math. Res. Inst., Oberwolfach, 1977)*, pages 53–67. Internat. Ser. Numer. Math., Vol. 40. Birkhäuser, Basel, 1978.
- [8] M. G. Cowling. Harmonic analysis on semigroups. *Ann. of Math. (2)*, 117(2):267–283, 1983.
- [9] Georges de Rham. *Differentiable manifolds*, volume 266 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984. Forms, currents, harmonic forms, Translated from the French by F. R. Smith, With an introduction by S. S. Chern.
- [10] U. Dinger. Weak type $(1, 1)$ estimates of the maximal function for the Laguerre semigroup in finite dimensions. *Rev. Mat. Iberoamericana*, 8(1):93–120, 1992.
- [11] O. Dragičević and A. Volberg. Bilinear embedding for real elliptic differential operators in divergence form with potentials. *J. Funct. Anal.*, 261(10):2816–2828, 2011.
- [12] L. Forzani, E. Sasso, and R. Scotto. Weak-type inequality for conjugate first order Riesz-Laguerre transforms. *J. Fourier Anal. Appl.*, 17(5):854–878, 2011.
- [13] L. Forzani, E. Sasso, and R. Scotto. Weak-type inequalities for higher order Riesz-Laguerre transforms. *J. Funct. Anal.*, 256(1):258–274, 2009.
- [14] J. García-Cuerva, G. Mauceri, S. Meda, P. Sjögren, and J. L. Torrea. Functional calculus for the Ornstein-Uhlenbeck operator. *J. Funct. Anal.*, 183(2):413–450, 2001.
- [15] P. Graczyk, J.-J. Loeb, I. A. López P., A. Nowak, and W. O. Urbina. Higher order Riesz transforms, fractional derivatives, and Sobolev spaces for Laguerre expansions. *J. Math. Pures Appl. (9)*, 84(3):375–405, 2005.

- [16] L. Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2008.
- [17] M. Gromov. Kähler hyperbolicity and L_2 -Hodge theory. *J. Differential Geom.*, 33(1):263–292, 1991.
- [18] C. E. Gutiérrez, A. Incognito, and J. L. Torrea. Riesz transforms, g -functions, and multipliers for the Laguerre semigroup. *Houston J. Math.*, 27(3):579–592, 2001.
- [19] W. Hebisch, G. Mauceri, and S. Meda. Holomorphy of spectral multipliers of the Ornstein-Uhlenbeck operator. *J. Funct. Anal.*, 210(1):101–124, 2004.
- [20] X.-D. Li. On the strong L^p -Hodge decomposition over complete Riemannian manifolds. *J. Funct. Anal.*, 257(11):3617–3646, 2009.
- [21] X.-D. Li. Riesz transforms on forms and L^p -Hodge decomposition on complete Riemannian manifolds. *Rev. Mat. Iberoam.*, 26(2):481–528, 2010.
- [22] X.-D. Li. On the weak L^p -Hodge decomposition and Beurling-Ahlfors transforms on complete Riemannian manifolds. *Probab. Theory Related Fields*, 150(1-2):111–144, 2011.
- [23] N. Lohoué and S. Mehdi. Estimates for the heat kernel on differential forms on Riemannian symmetric spaces and applications. *Asian J. Math.*, 14(4):529–580, 2010.
- [24] N. Lohoué. Estimation des projecteurs de de Rham Hodge de certaines variétés riemanniennes non compactes. *Math. Nachr.*, 279(3):272–298, 2006.
- [25] G. Mauceri, S. Meda, and P. Sjögren. Sharp estimates for the Ornstein-Uhlenbeck operator. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 3(3):447–480, 2004.
- [26] B. Muckenhoupt. Poisson integrals for Hermite and Laguerre expansions. *Trans. Amer. Math. Soc.*, 139:231–242, 1969.
- [27] B. Muckenhoupt. Conjugate functions for Laguerre expansions. *Trans. Amer. Math. Soc.*, 147:403–418, 1970.
- [28] F. Nazarov, S. Treil, and A. Volberg. The Bellman functions and two-weight inequalities for Haar multipliers. *J. Amer. Math. Soc.*, 12(4):909–928, 1999.
- [29] F. L. Nazarov and S. R. Treil. The hunt for a Bellman function: applications to estimates for singular integral operators and to other classical problems of harmonic analysis. *Algebra i Analiz*, 8(5):32–162, 1996.
- [30] A. Nowak. On Riesz transforms for Laguerre expansions. *J. Funct. Anal.*, 215(1):217–240, 2004.
- [31] A. Nowak and K. Stempak. On L^p -contractivity of Laguerre semigroups. *Illinois J. Math.*, 56(2):433–452, 2012.
- [32] E. Sasso. Functional calculus for the Laguerre operator. *Math. Z.*, 249(3):683–711, 2005.
- [33] E. Sasso. Spectral multipliers of Laplace transform type for the Laguerre operator. *Bull. Austral. Math. Soc.*, 69(2):255–266, 2004.
- [34] E. Sasso. Maximal operators for the holomorphic Laguerre semigroup. *Math. Scand.*, 97(2):235–265, 2005.
- [35] E. Sasso. Weak type estimates for Riesz-Laguerre transforms. *Bull. Austral. Math. Soc.*, 75(3):397–408, 2007.
- [36] E. M. Stein. *Topics in harmonic analysis related to the Littlewood-Paley theory*. Annals of Mathematics Studies, No. 63. Princeton University Press, Princeton, N.J., 1970.
- [37] R. S. Strichartz. Analysis of the Laplacian on the complete Riemannian manifold. *J. Funct. Anal.*, 52(1):48–79, 1983.
- [38] F. W. Warner. *Foundations of differentiable manifolds and Lie groups*, volume 94 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1983. Corrected reprint of the 1971 edition.
- [39] B. Wróbel. *Dimension free L^p estimates for Riesz transforms via an H^∞ joint functional calculus*, arXiv:1404.6921 [math.FA], 2014
- [40] A. Zettl. *Sturm-Liouville theory*, volume 121 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005.

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